Documentos de Trabajo 25

Information Provision in Competing Auctions

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First version: August 2010
This version: June 2011

Abstract

This paper studies the incentives faced by competing auctioneers who can provide information to prospective bidders about their valuations of the objects for sale. We consider a model in which two sellers running second-price auctions compete to attract potential bidders by releasing information about valuations before bidders select trading partners. Thus, bidders' participation decisions are modeled in ex-post terms which allows us to investigate the effect of information on the composition of the set of types who visit each seller. We derive a set of necessary and sufficient conditions that supports full information provision as the unique equilibrium of the game. This result holds even if the number of bidders is restricted to two, which contrasts with the findings of models with a single auctioneer where full information provision is never optimal. We also provide a characterization of information in terms of its strategic value to the sellers.

Keywords: Competing Auctions, Information Structures, Private Provision of Information.
JEL classification: C72, D44, D82

*This paper is part of my ongoing Ph.D. thesis in the Department of Economics at the University of British Columbia. I would like to thank Li Hao for detailed comments and constant encouragement during early stages of this project. This paper has been presented at Universidad Diego Portales, Universidad Técnica Federico Santa Maria, the 2010 Meeting of the Chilean Economic Association, the 2011 Spring Midwest Economic Theory Meeting, and the 2011 North American Summer Meeting of the Econometric Society. I am grateful for the insightful comments I received from these audiences. As usual, all remaining errors are my own responsibility.

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1 Introduction

There are many situations in which sellers have the ability to control the amount of information available to potential buyers. In auctions, sellers can decide what potential bidders learn about the objects for sale by choosing how much information about the items to release prior to commencing with the auction: while some sellers may decide to post full color pictures together with very detailed descriptions of the objects, some others may (deliberately) omit important details and post only basic information. If we think of buyer's valuation as made of two elements, one related to her personal preferences and another one related to the *quality* or characteristics of the item, then by providing more or less information sellers can influence the second of these elements. In this paper, we are interested in the problem of information provision in environments where sellers must compete for the attention of buyers. In particular, we would like to know how competition among sellers affects the incentives that a given seller faces when choosing how much information to provide to potential bidders.

While the problem of information provision by a single seller has received some attention in the literature (Ganuza and Penalva, 2004, 2006; Bergemann and Pesendorfer, 2007; Esso and Szentes, 2005), environments where several sellers compete for the attention of customers has received little (Forand, 2009). One reason that may explain this apparent lack of interest is the complexity inherent to models with competing sellers McAfee (1993); Peters and Severinov (1997); Burguet and Sakovics (1999); Virag (2010); Peters (2010) plus the fact that competition through information provision generates a complex trade off between market share and the cost of selling the object to bidders with better private information (Forand, 2009). However, interesting questions such as how valuable information become when used as an strategic tool or how the incentives to provide information are affected by the number and characteristics of potential buyers can only be addressed in models that explicitly incorporates competition among sellers. In this paper we aim at shedding some light on these and other equally interesting questions.

The model we develop is as follows. Two sellers with unit supply compete for the unit demands of $n \geq 2$ potential bidders by releasing information *before* bidders decide which auction they want to visit. We assume that sellers can let bidders learn their valuations perfectly or they can leave bidders completely uninformed, i.e., we assume that information has a *binary* rather than a continuous structure. Despite this binary structure, the fact that bidders select trading partners in ex-post rather than ex-ante terms allows us to capture the effect of information provision and competition on the composition of the pool of types who visit each seller. Vagstad (2007) has already considered this issue in a model where potential bidders must pay an entry cost before learning their values. In his model, early information induces *screening* of high-valuation bidders with the seller having to pay extra informational rents to all bidders. Since Vastag considers only one seller, bidders' outside option cannot be affected by information because a bidder who chooses not to participate simply earns some exogenously given rent. He finds that the auctioneer has too weak an incentive to produce early information because releasing information can drastically reduce entry, which translate into lower profits compared to what the seller could obtain by not releasing information. In contrast, we show that this need not be the acase when competition among sellers is explicitly taken into consideration. With competing sellers, bidders outside option is *endogenous* as the rent any given bidder expects in each auction depends on what *both* sellers are expected to do in terms of information provision. This has the potential to drastically change the incentives sellers face when deciding whether to provide information or not because they can now use information to affects not only entry but also the composition of the set of types who visit their auctions. One of the first attempts to recognize this issue corresponds to Forand (2009), where sellers compete by promising to reveal information to all bidders who commit to participating in their respective auctions. This way to model bidders' participation decisions considerably simplify the analysis (because participation decisions are taken in ex-ante rather than in ex-post terms) but it fails to capture one of the most interesting aspects of information provision in competign environments: the way in which information can be used to alter the pool of types that visits each auction[1]. Consequently, we model participation in ex-post terms, i.e., we let bidders choose trading partners after they have learned (if so) the actual value they assign to the items. Up to our knowledge, we constitute the first formal attempt to study the interaction between information provision and competition when bidders choose trading partners in ex-post terms, making it a novel contribution of our paper.

Our main findings are as follows. First, existence of an equilibrium in which sellers do not provide information depends on both the total number of potential bidders and their characteristics summarized by the distribution function of valuations. For small markets, i.e., markets in which the number of potential bidders is small, existence of this type of equilibrium requires the distribution of bidders' valuation to satisfy restrictions in terms of its location parameters (its mean) and in terms of its dispersion. Intuitively, what we need is to have a prior distribution that does not concentrate too much mass on its right tail (more precisely, we need that $F(\mu) > \frac{1}{2}$ holds, with μ the mean of F). This is necessary in order to avoid having too many bidders with relatively high valuations (with respect to the mean) for whom is atractive to visit the seller who provides information. When the distribution of valutions does not satisfy this condition, a given seller (say seller 1) who expects the other seller to choose an uninformative structure is able –by revealing information, to attract those bidders who value his item the most, leaving those with relatively low valuations out of his auction. In other words, against a competitor who is not expected to provide information, the seller who choose to inform potential bidders can affect the pool of types in such a way that only bidders with relatively high valuations are willing to visit his auction, which is sufficient to rule out equilibria without information provision. It is important to highlight the fact that the condition $F(\mu) \leq \frac{1}{2}$ suffices to rule equilibria without information provision regardless of the number of potential bidders, which contrats with the results of models with a single auctioneer and two potential bidders where information provision is never optimal for the seller.

Second, we derive a set of necessary and sufficient conditions for the game to support an equi-

[1] However, Forand's paper is more general in several other aspects as he considers competition in direct mechanisms while we restrict competition to auctions only.

librium in which both sellers provide information regardless of the number of bidders in the market. Roughly speaking, what these conditions do is to impose some restrictions on the location of the mean and the dispersion of the distribution of bidders' valuations. Restrictions on the location of μ are needed because information affects the expected traffic at each auction, and the magnitud of this traffic effect depends on the mass of types located to the right/left of the mean of F. Likewise, restrictions on the level of dispersion of F are needed because we have to handle the effect of information on the expected price. As providing information is akin to introducing heterogeneity in the valuations of bidders, too much dispersion may mean too much heterogeneity to make the expected price sufficiently attractive to the seller. However, we show that for markets with three or more bidders the price that a given seller expects if he and his competitor provide information is never less than the price this seller would obtain if he decided not to supply information to the market. When the number of bidders is exactly equal to two, restrictions on the degree of dispersion of the distribution of valuations are needed in order to ensure that the price when both sellers provide information is no less than the price when only one of them chooses an informative structure. We also show that having a convex distribution of valuations functions guarantees not only existence but also uniqueness of an equilibrium in which both sellers provide information.

Finally, we attempt a characterization of information in terms of its strategic value for sellers. We show that, in the case in which information does not affect expected traffic, the incentives to provide information faced by any given seller are weaker in environments where his competitor is also expected to choose an informative structure. That is, the value that a seller assigns to information decreases with the decision of his competitor to provide information, i.e., information behaves as an *strategic substitute*. Intuitively, information can be considered as an strategic substitute because providing information against a competitor that also does so, does not preclude the visit of bidders with low valuations, whereas providing information againts a competitor that is expected not to provide does preclude these visits. This, in turn, occurs because provision of information in environments where only one seller is expected to do so translates into a truncation on the lower limit of the support of valuations. On the contrary, providing information againts a competitor who also does so only reallocates probability mass without affecting the support of the distribution of valuations. Although we have not been able to fully extend this analysis to more general cases (i.e. cases in which information also affects expected traffic), we do provide a set of necessary (sufficient) conditions for information to be an strategic complement (substitute) in markets with a small number of potential bidders.

The rest of the paper is organized as follows. We begin with an example intended to motivate the types of issues we are interested in. Next section is used to outline the fundamentals of our model. The following two sections provide the characterization of the equilibrium set of our game. We conclude the paper with some final comments and conclusions.

1.1 An Example

An example may help to motivate the types of issues that concern us on this paper. Suppose that two sellers with unit supply (say of a car) are trying to trade with two buyers with unit demand[2]. Conditional on trading, the goods are allocated via second-price sealed bid auctions without reserve price. Cars are identical in all aspects (brand, engine, etc) except for the color. Buyers have preference over colors, but these preferences are private information. A given buyer $i = 1, 2$ either likes or dislikes a given car depending on whether its color matches the buyer's favorite one. If a given car is not a match to buyer i's tastes, she assigns a value of $\theta_i^j = L = 0$ to it, whereas if this car is a match to her tastes she assigns it a value of $\theta_i^j = H = 1$. Prior to any interaction with sellers, buyers are unaware of the true color of the cars. However, they know that the probability of having a mismatch is equal to $\frac{1}{2}$, so that the probability of having a match is also equal to $\frac{1}{2}$. It is assumed to be common knowledge that having a match with seller j is independent of having a match with seller $-j$ (i.e., valuations are independent between goods and across buyers). Sellers can help buyers to reduce this uncertainty by posting an ad that includes a (true) picture of the the color of his car. That is, by posting an ad seller j perfectly informs both bidders about the true color of his car. Since bidders' preferred colors are privately known, how much each buyer values a given car is private information. The main question we want to ask is whether sellers have enough incentives to post ads and how these incentives depends on the competition between sellers.

Consider the subgame in which neither seller posts ads. In this subgame, each buyer sees both cars as (ex-ante) identical. If we further assume that buyers cannot coordinate in their visiting decisions (an assumption that may not be suitable with only two buyers but that it is relevant when n is big), then in the continuation equilibrium each buyer visits seller j with probability $\frac{1}{2}$. Given this continuation play, and the fact that each seller expects a positive price if and only if both buyers visits (cars are auctioned off using second-price auctions with no reserve prices), the price each seller expects when no ad is posted is equal to the average valuation[3], i.e., equal to $\frac{1}{2}$. Consequently, each seller's expected profit is equal to $V^{\sim} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Consider now the subgame in which one seller (say seller 1) decides to post an ad while seller 2 does not. By posting an ad, seller 1 perfectly informs *both* buyers about their true valuations of this item and thus, buyers' participation decisions are taken conditional on their respective valuation of the car offered by seller 1. It is not difficult to check that an optimal play is for each buyer to visit seller 1 with probability one whenever $\theta_i^1 = H$ and to visit this seller with probability zero otherwise[4]. Under this continuation play seller 1 expects

[2]In what follows, we reserve male pronouns to refer to sellers and female ones to refer to bidders/buyers.

[3]We are assuming that bidders bid their (expected) valuation truthfully.

[4]First, notice that regardless of whether the other buyer comes or not, a buyer who has learned that her valuation is $\theta^1 = L$ expects no surplus in seller 1's auction while she still has a chance of pocketing a positive (expected) payoff by attending auction 2 (because her valuation for item 2 is equal to $\frac{1}{2}$). Hence, attending auction 2 with probability one when $\theta^1 = L$ is optimal for this type. As for a buyer (say buyer 1) who has learned that $\theta^1 = H$, if she expects the other buyer to follow this participation rule then her expected payoff in auction 1 is $\frac{1}{2}$ because she wins the good for free whenever buyer 2's valuation is L (an event with probability $\frac{1}{2}$). If she visited seller 2 her expected payoff would equal $\frac{1}{4}$ because this buyer expects to win item 2 for free (an object worth $\frac{1}{2}$ to her) if only if the other bidder does

a profit of $V^A = \frac{1}{4}$ because she gets a positive price (equal to $H = 1$) if and only if both buyers visit him (an event with probability $\frac{1}{4}$). Comparing V^A with $V^{\sim}$ it is clear that posting no ad cannot constitute an equilibrium. In fact, it is straightforward to check that this game admits a unique (symmetric) equilibrium in which both sellers post ads.

One interesting aspect of the previous example is the fact that the deviation for seller 1 is profitable because providing information allows him to attract bidders with high valuations. As we mentioned in the introduction, Vagstad (2007) has studied this problem in a model with a single seller in which information provision induces screening of high-valuation bidders. However, in his model bidders' outside option is exogenous which makes the auctioneer's incentives to release (early) information too weak. In light of our example, it appears that Vastag's conclusion depends on the existence of a single seller in the market. When two sellers compete for the attention of buyers, bidders' participation decisions depend on how much rent they expect to obtain in each auction, making each buyer's outside option an endogenous object of the model. Our example suggests that providing information decrease the value of the bidders' outside options relative to what they could obtain in a model where only one seller operates the market.

2 The Model

Consider an economy in which trade takes place using second-price sealed bid auctions with no reserve prices. The economy is populated by two risk-neutral sellers (seller 1 and seller 2) with unit supply and $n \geq 2$ risk neutral buyers with unit demands indexed by $i \in N = \{1, \cdots, n\}$. Bidder i's true vector of valuations is $v_i = (v_{i,1}, v_{i,2})$, where $v_{i,j}$ represents bidder i's valuation of good $j = 1, 2$. The vector v_i is a tuple of jointly independent random variables, with $v_{i,j}$ independently drawn from a common probability distribution $F(\cdot)$ with continuously differentiable and positive density $f(v) > 0$, and full support $[0, 1]$. Prior to any interaction with sellers, bidders are uncertain about how well each item matches their preferences. Sellers can help reduce this uncertainty by choosing to reveal information that makes bidders fully aware of their true (private) valuations. Alternatively, a seller can choose not to provide information in which case bidders remain unaware of their true valuation of this item. For simplicity, we refer to the choice of a perfectly information structure by seller j as $p_j = 1$, and let $p_j = 0$ stands for seller j's choice of a perfectly uninformative structure. Occasionally, we refer to seller's choice of information structures as the degree or level of informativeness.

The game we study begins with sellers simultaneously announcing some levels of informativeness that become common knowledge immediately after chosen. Next, Nature draws a pair of independent signals $(s_{i,1}, s_{i,2}) \in [0, 1]^2$ using the common prior distribution F, which she privately communicates to bidder i, $i = 1, \cdots, n$. Then, bidder i either perfectly learns her valuation of good j (if $p_j = 1$ then $s_{i,j} = v_{i,j}$) or remains uncertain (if $p_j = 0$ then $s_{i,j}$ is pure noise coming from the same distribution F) about it, in which case her valuation is simply equal to the mean μ of the distribution $F(\cdot)$. After updating has taken place, each bidder independently and simultaneously decides whether to

not visit this same seller, which happens with probability $\frac{1}{2}$.

participate in one and only one auction. Next, all bidders participating in seller j's auction learn the actual number of participants and then submit bids. Seller j collects these bids and awards the good using a second price sealed-bid auction with reserve price set equal to zero. Normalizing reserve prices to zero is made in order to keep the focus exclusively on the information provision side of the problem without having to worry about the interaction between reserve prices and information provision.

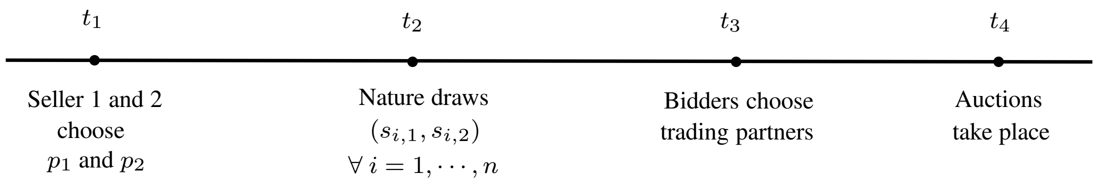

Figure 1: Timing of the Game

3 Equilibrium Analysis

From our previous discussion, we know that by the time of submitting a bid to seller j either all or none of the bidders will be (privately) informed about their valuations of j's item. Irrespective of what their estimates are, it is well known that bidding truthfully forms a Bayesian-Nash equilibrium of the bidding game. In order to keep things as simple as possible, we follow the literature of competing auctions and assume that each bidder bids truthfully whatever estimate she may have. What this simplification seeks is to reduce buyers' problem to her decision about which seller to visit. Moreover, as participating in some auction always yields a nonnegative payoff (when bidding is truthful), it is without loss of generality to treat the decision of not participation as a a non-serious bid. This allows us to define a strategy for buyer i as a mapping $\pi_i : \{0,1\}^2 \times [0,1]^2 \rightarrow [0,1]$ specifying a probability with which this buyer visits seller 1 (the probability of visiting seller 2 is $1 - \pi_i$) as a function of the possible choices of sellers' informativeness levels and her information at $t = 2$.

We restrict attention to (Perfect Bayesian) equilibria in which bidders use *symmetric participation rules.* A participation rule is symmetric if two bidders with the same vector of estimates visit seller 1 with the same probability, $\pi_i(\cdot) = \pi_k(\cdot) \equiv \pi(\cdot)$, $i \neq k \in \{1, 2\}$.

3.1 Bidder's continuation game

We begin our analysis with the characterization of the symmetric participation rule used by bidders to select trading partners in equilibrium. Since the degree of informativeness can take only two values (zero or one), there are three subgames we need to consider: (i) the subgame in which both sellers announce uninformative structures ($p_1 = p_2 = 0$); (ii) the subgame in which one seller chooses an uninformative structure while the other chooses a perfectly informative one ($p_j = 1; p_{-j} = 0$); (iii) the subgame in which both sellers announce perfectly informative structures ($p_1 = p_2 = 1$).

Case (i) is the simplest. Since both sellers are choosing perfectly uninformative structures, signals are irrelevant and both items are worth the same for every bidder. This together with the fact that bidders do not coordinate on their visiting decisions means that in equilibrium any given bidder visits each seller with probability $\frac{1}{2}$. Hence, $\pi(\cdot)$ must satisfy $\pi((0,0),(s_1,s_2)) = \frac{1}{2}$ for all $(s_1,s_2) \in [0,1]^2$.

Case (ii) is slightly more involved. Without loss of generality, suppose that seller 1 chooses $p_1 = 1$ while seller 2 chooses $p_2 = 0$. Since signals coming from seller 2 are pure noise (distributed according to F), we have $v_{i,2} = \mu$ for all $i = 1, \cdots, n$. Suppose that all bidders except bidder 1 use the same participation rule $\pi(\cdot)$. From bidder 1's perspective, the probability of winning the item at seller 2's auction is equal to $\left(\int_0^1 \pi(v) f(v) dv\right)^{n-1}$ because bidder 1 expects any other bidder coming to auction 2 to bid the exact same valuation (μ) and bidder 1 wins the item only if she is the unique participant in the auction. Therefore, bidder 1's ex-ante expected payoff if she decides to visit seller 2 is:

$$U_2 \quad = \quad \mu \left(\int_0^1 \pi(v) f(v) dv \right)^{n-1}$$

Alternatively, as seller 1 is choosing $p = 1$ every bidder uses her signal to update her valuation of this item, which generates heterogeneity in the distribution of valuations. Let $q_1(v)$ be the probability with which type v of bidder 1 wins the item at seller 1's auction. Since this type of bidder 1 wins this item when either all other bidders go to seller 2 or, conditional on visiting seller 1 they do so with valuations less than his (McAfee, 1993), $q_1(v)$ is given by:

$$q_1(v) \quad = \quad \left(1 - \int_v^1 \pi(z) f(z) dz\right)^{n-1}$$

which yields the ex-ante expected payoff of type v at auction 1:

$$U_1(v) = U_1(\underline{v}) + \int_{\underline{v}}^{v} q_1(z) dz$$

where $\underline{v} = \inf\{v : \pi(v) > 0\}$ is the lowest type willing to participate in auction 1 with positive probability. Unsurprisingly, the rest of the characterization of the equilibrium participation rule employed by bidders in this subgame can be done using the idea of cutoff strategies as much as it is done in auction models with costly participation (e.g. Tan and Yilankaya, 2006).

Lemma 3.1. *Consider the continuation game arising after seller 1 and seller 2 have announced $p_1 = 1$ and $p_2 = 0$. Then, there exists a unique cutoff valuation $v^* \in (0,1)$ such that any given bidder with valuation v visits seller 1 with probability one whenever $v \geq v^*$, and visits seller 2 with probability one whenever $v < v^*$. The value of $v^* \in (0,1)$ is uniquely defined by:*

$$v^* F(v^*)^{n-1} = (1 - F(v^*))^{n-1} \mu \tag{1}$$

Proof. All proofs are relegated to the Appendix. □

Simple inspection of Eq. (1) reveals that the exact value of v^* depends on the characteristics of the distribution function F. In particular, notice that v^* and μ are related through the location that μ occupies with respect to the median of F. For instance, if F is symmetric then the value of v^* coincides with that of μ (symmetric distribution functions satisfies $F(\mu) = \frac{1}{2}$). Intuitively, this is due to the fact that symmetric distribution put the same mass to the left and to the right of the mean and thus, the probability that a given bidder draws a valuation lower or higher than μ is the same. The following lemma formalizes and extends this intuition by providing a set of necessary and sufficient conditions in terms of two location parameters of F, its mean and its median, that allows us to pin down the exact value of v^*.

Lemma 3.2. *Let v^* be implictly defined by Eq. (1), and v^m be the median of $F(\cdot)$, i.e. the value of $v \in [0,1]$ that satisfies $F(v^m) = \frac{1}{2}$. Then, v^* satisfies $\mu \leq v^* \leq v^m$ if and only if $F(\mu) \leq \frac{1}{2}$, and $v^m < v^* < \mu$ if and only if $F(\mu) > \frac{1}{2}$.*

The remaining case (iii) is perhaps the more interesting one. Consider the continuation game in which both sellers provide information to bidders (i.e., the subgame that arises after a history such that $p_1 = p_2 = 1$). Since valuations are stochastically independent random variables privately known by bidders, characterizing the continuation play requires the production of a bidimensional object mapping *pair* of valuations into visiting probabilities. In principle, this characterization is technically challenging. However, the fact that information structures are binary and reserve prices are both fixed at zero simplifies considerably the analysis. Similar to Troncoso-Valverde (2011), the idea is to redirect the analysis toward the characterization of *cutoff functions* used by bidders to select trading partners. Roughly speaking, a cutoff function is a mapping $\rho : [0,1] \rightarrow [0,1]$ specifying a threshold value $\rho(v_2) \in [0,1]$ for every $v_2 \in [0,1]$ such that any given bidder whose valuations are (v_1, v_2) selects seller 1 as her trading partner with probability one if $v_1 \geq \rho(v_2)$, else she visits seller 2 for sure. This idea (of bidders using cutoff functions) has been employed in auction models with stochastic valuations and stochastic entry costs (Green and Laffont, 1984; Lu, 2006). The main difference between these models and ours is the exogeneity/endogeneity of the bidder's cost of participating in an auction. In the former models the cost a bidder has to pay in order participate in the auction is given by the (privately) observed realization of a random variable. In contrasts, in our model the cost of participating in, say, auction 1 is equal to the the rent this bidder would obtain had she participated in auction 2 instead. Obviously, this foregone rent is an endogenous object and hence, our characterization can be viewed as a generalization of cutoff strategies to environments where participation costs are endogenous.

As a preliminary step, we present a result that gives support to our characterization of equilibria using cutoff functions. It shows that any given equilibrium play in this continuation game can be replicated by an action in which bidders choose trading partners using cutoff functions. Thus, no matter how complex the equilibrium participation rule looks like, we can always replicate the equilibrium play after histories in which $p_1 = p_2 = 1$ by assuming that bidders select sellers using a common cutoff function[5].

[5]It is straightforward to show that this results holds no matter whether we focus on symmetric or asymmetric

Lemma 3.3. *Consider the continuation game arising after a history in which* $p_1 = p_2 = 1$. *Then, any equilibrium of this continuation game can be described through a continuous and increasing cutoff function* $\rho : [0,1] \to [0,1]$ *with the property that any bidder with valuations* (v_1, v_2) *visits seller 1 with probability one if* $v_1 \geq \rho(v_2)$, *and visits seller 2 with probability one if* $v_1 < \rho(v_2)$.

The importance of Lemma (3.3) relies on the fact that questions about existence and uniqueness of continuation equilibria can be addressed by looking for an increasing and continuous function with the properties described in this lemma. Since valuations are independent accross bidders and items, the probability with which any bidder visits seller 1 when bidders use the cutoff function ρ is[6]:

$$\begin{aligned} q &= \int_0^1 \int_{c(\xi)}^1 dF(\zeta)dF(\xi) \\ &= 1 - \int_0^1 F(c(\xi))f(\xi)d\xi \end{aligned} \tag{2}$$

Without loss of generality, take type $(\rho(v), v)$ of bidder 1 and suppose that this type chooses to participate in auction 1. Let $Q_1(x)$ be the probability with which a bidder with valuation x of item 1 wins the object. Following McAfee (1993), $Q_1(x)$ is equal to:

$$Q_1(x) = \left[1 - q + \int_{\{\xi:\rho(\xi)\leq x\}} \{F(x) - F(\rho(\xi))\} f(\xi)d\xi\right]^{n-1} \tag{3}$$

The payoff that type $(\rho(v), v)$ expects when she participates in auction 1 depends on the degree of competition in this auction: If this type is alone then she obtains the object for free, else her payoff depend on the highest bid among those who participate in this auction. Thus, the payoff that type $(\rho(v), v)$ expects when she participates in auction 1 with positive probability is[7]:

$$U_1(\rho(v)) = \rho(v)(1-q)^{n-1} + \int_0^{\rho(v)} (\rho(v) - z)dQ_1(z) \tag{4}$$

The first term in Eq. (4) corresponds to the payoff type $(\rho(v), v)$ obtains when she is alone at auction 1. The second term is her payoff when she faces competition from other bidders but she turns out to be the highest bidder in the auction. Notice that we have to integrate $(\rho(v) - z)$ between zero and $\rho(v)$ because the lowest type willing to participate in auction 1 with positive probability is type $(0, 0)$. Alternatively, if type $(\rho(v), v)$ chooses to participate in auction 2 with positive probability, her expected payoff would be:

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equilibria of this continuation game. Of course, if we look at asymmetric equilibria our characterization would consist in the production of a st of cutoff functions $\{\rho_i\}_{i=1}^n$ such that bidder i uses the function ρ_i to select her trading partner.

[6] Notice that unless $\rho(\cdot) \equiv 1$ or $\rho(\cdot) \equiv 0$, q must lie strictly between zero and one. However, from the proof lemma (3.3) we know that neither of these cases is possible. Therefore, q must always lie strictly between zero and one.

[7] We supress v_j from the arguments of U_{-j} whenever there is no risk of confussion.

$$U_2(v) \quad = vq^{n-1} + \quad \int_0^v (v - z) dQ_2(z) \tag{5}$$

where,

$$Q_2(z) = \left[q + \int_0^z F(c(\xi)) f(\xi) d\xi \right]^{n-1} \tag{6}$$

is the probability that a bidder with valuation of item 2 equal to z wins the item in auction 2. By construction, type $(\rho(v), v)$ must be indifferent about which seller to visit and hence, it must be true that her payoff is the same regardless of which seller she actually visits. Simple manipulation of equations (4) and (5) gives the following *indifference condition*:

$$\int_0^{\rho(v)} Q_1(z) dz = \int_0^v Q_2(z) dz \tag{7}$$

which must hold for all $v \in [0, 1]$. It is straightforward to see that type (v_1, v_2) of bidder 1 expects a strictly higher (lower) payoff than that of type $(\rho(v_2), v_2)$ whenever $v_1 \geq \rho(v_2)$ $(v_1 < \rho(v_2))$. Consequently, choosing trading partners according to $\rho(\cdot)$ does constitute an equilibrium play in this continuation game. What it remains to be shown is the existence of a solution to Eq. (7). The following result takes care of this problem and also provides a close-form solution to (7), which is thank to the fact that reserve prices are fixed and normalize to zero. Unsurprisingly, the solution to Eq. (7) corresponds to the identity function.

Lemma 3.4. *Consider the continuation game following a history in which sellers announce $p_1 = p_2 = 1$. Then, there exists a unique function $\rho(v) = v$ such that any given bidder whose vector of valuations is (v_1, v_2) visits seller 1 (seller 2) with probability one (zero) whenever $v_1 \geq \rho(v_2) = v_2$ $(v_1 < \rho(v_2) = v_2)$, for all $(v_1, v_2) \in [0, 1]^2$.*

Intuitively, a bidder whose valuations are the same should expect the same rent no matter which auction she visits because sellers are ex-ante identical (both charge the same reserve price) and everybody uses the same participation rule. Since expected rents and valuations are in a one-to-one relation (because rents are strictly increasing in the corresponding valuation), a bidder whose valuation of item 1 is greater than that of item 2 should participate in auction 1 with probability one. What it is not so obvious is the uniqueness of this equilibrium within the class of symmetric continuation equilibria of the game.

We close this section with a proposition that summarizes our discussion so far.

Proposition 3.1. *Consider the continuation game arising after a history in which sellers announce some pair $(p_1, p_2) \in \{0, 1\}^2$. Then, the following participation rule constitutes the unique continuation* *equilibrium within the class of symmetric continuation equilibria:*

$$\pi(p_1, p_2, s_1, s_2) = \begin{cases} \frac{1}{2} & \textit{if } p_1 = p_2 = 0 \\ 1 & \textit{if } v_1 \geq v^*,\ p_1 \cdot p_2 = 0,\ \textit{and } p_1 = 1 \\ 0 & \textit{if } v_1 < v^*,\ p_1 \cdot p_2 = 0,\ \textit{and } p_1 = 1 \\ 1 & \textit{if } v_1 \geq v_2,\ \textit{and } p_1 \cdot p_2 = 1 \\ 0 & \textit{if } v_1 < v_2,\ \textit{and } p_1 \cdot p_2 = 1 \end{cases} \tag{8}$$

where the unique value of v^* is given by Eq. (1)*.*

3.2 Equilibria of the Information Provision Game

Consistent with subgame perfection, we now proceed with the analysis of sellers' choices of information structures made in the first stage of the game, conditional on the continuation game being played according to the participation rule given by Eq. (8) (recall that bidding is assumed to be truthfull).

3.2.1 Equilibrium without Provision of Information

To begin with, suppose that both sellers choose completely uninformative information structures, i.e., $p_1 = p_2 = 0$. Then, in the continuation game bidders randomize with equal probability between seller 1 and seller 2. Since both sellers set reservation prices equal to zero, the only chance for either of them to obtain a positive payoff is to have at least two visitors, in which case the seller receives a price equal to μ. Let $V_j(p_j, p_{-j})$ be the expected payoff of seller j when he chooses a degree of informativeness p_j and his competitor chooses p_{-j}. As the probability that two or less bidders visit seller 1 is equal to $\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n$, seller 1's payoff when both he and his competitor chooses completely uninformative structures ($p_1 = p_2 = 0$) is given by:

$$V_1(0,0) = \left[1 - (1+n)\left(\frac{1}{2}\right)^n\right]\mu \tag{9}$$

Now suppose that, while seller 2 keeps choosing $p_2 = 0$, seller 1 decides to announce a perfectly informative structure ($p_1 = 1$). In this case, the continuation equilibrium features bidders choosing seller 1 as their trading partner based on whether their privately known valuations are above a certain threshold value defined by Eq. (1). Similar to the previous case, any given seller needs at least two visitors in order to receive a positive payoff. However, the price expected at each auction is different as information structures differs in each site. For seller 1, the fact that bidders perfectly know their valuations affects the price he expects to receive in two different ways: (i) more information enhances the social surplus by improving the match between bidders and the seller; (ii) more information increases the rents bidders are expected to receive if they participate in this auction. While the first of these effects acts favoring a higher price, the second one tends to reduce it. However, the fact that only bidders with valuations above the threshold value v^* select auction 1 tends to alleviate the second effect because it is not necessary to offer informational rents to every bidder but only to those with valuations above v^*. Let $T(k, v^*)$ be the price seller 1 expects to receive when he is matched with exactly k bidders. As only bidders with valuations above v^* visit seller 1, define $H_{v^*}(v, k)$ as the distribution of the second order statistics conditional on being $k \geq 2$ visitors at seller 1's auction, when the distribution of types is given by the truncation of $F(\cdot)$ from below with truncation point v^*. Then, the (ex-ante) expected payoff of seller 1 when $p_1 = 1$ and $p_2 = 0$ is:

$$V_1(1,0) = \sum_{k=2}^{n} \binom{n}{k} (1 - F(v^*))^k F(v^*)^{n-k} T(k, v^*) \tag{10}$$

where $T(k, v^*) = \int_{v^*}^{1} v dH_{v^*}(v, k)$. Clearly, the payoff function (10) depends on the value of the truncation point v^* through its effect on traffic $(1 - F(v^*))$ and price $(T(k, v^*))$. Intuitively, we should expect that the more mass is allocated to the right tail of F the more likely is that both traffic and price work in favor of information provision (i.e., $V_1(1,0) > V_1(0,0)$). Indeed, when F allocates more probability mass on its right tail, bidders are more likely to draw relatively high valuations, which increase both the probability with which they visit seller 1, and the price they are willing to pay for the item. According to lemma (3.2), if F satisfies $F(\mu) \leq \frac{1}{2}$ then the value of v^* is located to the right of μ, which is sufficient to guarantee that bidders visit seller 1 more often than what they would do if $p_1 = 0$. Moreover, if $F(\mu) \leq \frac{1}{2}$ holds then the price the winner is expected to pay lies strictly above μ. On the other hand, when $F(\mu) > \frac{1}{2}$ bidders come less often than what they would do if $p_1 = 0$ and there is a small chance that the price falls short of μ.

Lemma 3.5. *Consider the continuation game in which $p_1 = 1$ and $p_2 = 0$. Let $T(k, v^*)$ be the price expected by seller 1 conditional on being exactly $k = 2, 3, \cdots, n$ visitors at seller 1's auction, when the distribution of types is given by the truncation of F from below, with truncation point equal to the cutoff value v^*. Then, $F(\mu) \leq 1/2$ implies $T(k, v^*) > \mu$ for all $(k, v^*) \in \{2, \cdots, n\} \times (0, 1)$.*

As we already mention, providing information against a competitor who is not providing produces two competing effects on the expected price: firstly, it increases the expected price because social surplus is higher, and secondly, it reduces the expected price because bidders' informational rents are higher. However, the fact that only a subset of types chooses to visit seller 1's auction alleviates the second effect because it decreases the amount of rents that the seller has to offer in order to induce truth-telling.

Aside from affecting the expected price, provision of inormation also affects the expected traffic. According to Proposition (3.1), the probability with which any given bidder visits seller 1 when $p_1 = 1$ and $p_2 = 0$ is equal to the probability that v_1 be greater than v^*, i.e., $1 - F(v^*)$, whereas the probability of getting a visitor when $p_1 = p_2 = 0$ is $\frac{1}{2}$. Therefore, providing information against a competitor who does not *increases* (does not decrease) the expected traffic whenever $F(v^*) \leq \frac{1}{2}$. Thus, $F(v^*) > \frac{1}{2}$ must be necessary for $p_1 = 0$ to be a best response to $p_2 = 0$. However, this condition might not be sufficient because lemma 3.5 ensures that the expected price is above μ, regardless of the shape of F. Hence, in order to have an equilibrium without provision of information we must have a traffic effect sufficiently strong to offset the positive effect of price on seller's profit. The next proposition takes care of this discussion and provides a set of necessary and sufficient conditions for the existence of an equilibrium such that $p_1 = p_2 = 0$.

Proposition 3.2. *Consider the information provision game played between sellers when bidders select trading partners using the participation rule given by* (8)*, and subsequently bid their (interim) valuations truthfully. Let* v^* *be the unique solution to* $v^* F(v^*)^{n-1} - \mu(1 - F(v^*))^{n-1} = 0$*. Then,* $F(v^*) > \frac{1}{2}$ *is necessary and* $F(v^*) > \frac{3}{4}$ *is sufficient for the existence of some* n *such that both sellers not providing information (i.e., such that* $p_1 = p_2 = 0$*) is an equilibrium of the game.*

Observe that in order for $p_1 = 0$ to be a best response to $p_2 = 0$ it is necessary that the mean of F be located to the right of its median (lemma 3.2), which in general occurs when F is right-skewed. This should not be surprising at all as right-skewed distribution functions tend to put more mass on its left tail, making more likely that bidders draw low valuations. Proposition (3.2) shows that the probability with which bidders visit seller 1 cannot be greater than $\frac{1}{4}$ for this to be the case. Moreover, this Proposition suggests that an equilibrium in which sellers choose uninformative structures is more likely to exist the smaller the market, where small should be understood in terms of the number of potential customers. Indeed, it is not difficult to show (e.g. Ganuza and Penalva, 2004) that as n grows large the expected value of the second order statistics tends to one making the effect of information on the expected price too strong to be offset by the traffic effect.

3.3 Equilibrium with Provision of Information

The remaining candidate for a symmetric equilibrium of our game is the one in which both sellers announce informative structures. Suppose that seller 1 and seller 2 announce $p_1 = 1$ and $p_2 = 1$ respectively. According to Proposition (3.1), in the continuation game bidders with valuations (v_1, v_2) choose seller 1 with probability one whenever $v_1 \geq v_2$ else they choose seller 2 for sure. Hence, the distribution of valuations that seller 1 expects to see in his auction is given by:

$$\begin{aligned} \tilde{G}(x) : &= \frac{\int_0^x [F(x) - F(\xi)] f(\xi) d\xi}{1 - \int_0^1 F(\xi) f(\xi) d\xi} \\ &= F^2(x) \end{aligned}$$

Observe that as bidders select trading partners based on which valuation is greater, any bidder with valuation $v_2 = \epsilon$, with ϵ small but positive, will pick seller 1 so long as v_1 is slightly above ϵ. This means that a seller who provides information againts a competitor who also does so, cannot avoid having bidders with arbitrarily low valuations in his auction. Let $\tilde{T}(k) = \int_0^1 v d\tilde{H}(v)$ be the expected value of the second order statistics when seller 1 is matched with exactly $k \geq 2$ bidders and the underlying distribution of valuations is $F^2(x)$. Seller 1's expected profit when $p_1 = p_2 = 1$ is:

$$V_1(1,1) = \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n \tilde{T}(k) \tag{11}$$

because the probability with which each bidder visits seller 1 is $q = \int_0^1 F(v)f(v)dv = \frac{1}{2}$. Alternatively, if seller 1 chose $p_1 = 0$ instead of $p_1 = 1$ (while seller 2 keeps choosing $p_2 = 1$), then lemma (3.1) tells us that a bidder with valuation v_2 visits seller 1 with probability one when v_2 is lower than the cutoff value given by Eq. (1) (and submits a bid equal to μ). Thus, the ex-ante profit of seller 1 when $p_1 = 0$ and $p_2 = 1$ is:

$$\begin{aligned} V_1(0,1) &= \sum_{k=2}^{n} \binom{n}{k} [F(v^*)]^k [1 - F(v^*)]^{n-k} \mu \\ &= \mu \left[1 - (1 - F(v^*))^n - nF(v^*)(1 - F(v^*))^{n-1}\right] \end{aligned}$$

because seller 1 receives a price equal to μ provided that at least two bidders participate in his auction. Comparing these two expressions we can identify how the traffic and price effects impact on seller 1's profit. For the traffic effect, seller 1's profit will vary according to the location of v^* with respect to the location of μ. Thus, the traffic effect works in favor of information provision whenever $F(v^*) \leq \frac{1}{2}$ (because in this case the visiting probability falls below $\frac{1}{2}$, which is the probability of visit seller 1 when he sets $p_1 = 1$), and against it whenever $F(v^*) > \frac{1}{2}$. From Lemma (3.2), $F(v^*) \leq \frac{1}{2}$ is equivalent to $F(\mu) \leq \frac{1}{2}$, which is more likely to be satisfied the more left-skewed $F(\cdot)$ is.

Respect to the price effect, whether information provision increases or decreases the expected price depends on the distribution function $\tilde{G}(x)$ as well as on the number of potential bidders. As we know, providing information is akin to introducing heterogeneity in the valuations of bidders, and one way to measure the degree of heterogeneity of $\tilde{G}(x)$ is to look at the degree of dispersion embedded in it. Moreover, how heterogeneity affects the expected price also depends on the number of bidders willing to participate in the auction. In fact, it is reasonable to expect that the higher the number of potential bidders the smaller the effect of heterogeneity on price because of the fiercer competition among bidders. To put things on a more concrete level, suppose that the number of potential bidders is set to two ($n = 2$). Conditional on both bidders visiting seller 1 (otherwise the price is zero), the price seller 1 expects is $\tilde{T} = \mathbb{E}[\min\{v_{1,1}, v_{2,1}\}]$, where $v_{i,1}$ is bidder i's valuation of item 1. By Jensen's inequality, we have that $\tilde{T} = \mathbb{E}[\min\{v_{1,1}, v_{2,1}\}] \leq \min[\tilde{\mu}, \tilde{\mu}] = \tilde{\mu}$, where $\tilde{\mu}$ is the mean of $\tilde{G}(x)$. Simple comparison of $\tilde{G}(x)$ and $F(x)$ tells us that $\tilde{G}(x)$ first-order stochastically dominates $F(x)$ and therefore, $\tilde{\mu}$ cannot be less than $\mu = \int_0^1 x dF(x)$. Therefore, without additional structure, it is hard to say whether $\tilde{T}$ would be greater than or less than μ, at least when $n = 2$. Our next result provides this additional structure: it says that if F satisfies a condition related to the distribution of its second order statistics then we can say something definitive with respect to the order of these prices when $n = 2$.

Lemma 3.6. *Let $\tilde{T}$ be the expected price prevailing in auction 1 when $p_1 = p_2 = 1$ and seller 1 is matched with exactly two bidders. Define*

$$\Gamma(F) = \int_0^1 \left\{F(x) + F^4(x) - 2F^2(x)\right\} dx$$

Then, $\Gamma(F) \geq (<) 0$ if and only if $\tilde{T} \geq (<) \mu$.

The reason why we need to impose a restriction on the set of admisible distribution when $n = 2$ if we want to have a price greater than the mean of F, is the lack of competition generated when only two bidders are willing to participate in the auctions. Since the chance of meeting each other in the same auction is low, each bidder is able to secure more rents in terms of participation as the value of their 'outside option' is also larger the lower the number of potential bidders.

Aside from affecting price, providing information against a competitor who also does so affects the expected traffic. From our previous discussion we know that the magnitud of this effect depends on the *location* of the cutoff value v^* relative to μ. This, together with lemmas (3.6) and (**??**), gives rise to the following Proposition, which provides a set of necessary and sufficient conditions for the existence of an equilibrium in which both sellers provide information.

Proposition 3.3. *Consider the information provision game played between sellers when bidders select trading partners using the participation rule given by* (8)*, and subsequently bid their (interim) valuations truthfully. Define* $\Gamma(F) = \int_0^1 \{F(x) + F^4(x) - 2F^2(x)\}dx$*. Then,* $\left(F(\mu) \leq \frac{1}{2} \text{ or } \Gamma(F) \geq 0\right)$ *is necessary whereas* $\left(F(\mu) \leq \frac{1}{2} \text{ and } \Gamma(F) \geq 0\right)$ *is sufficient for the existence of an equilibrium in which both sellers reveal information for all* $n \geq 2$*.*

It is interesting to notice that the set of sufficient conditions given in Proposition (3.3) guarantees not only existence but also uniqueness of an equilibrium. As $F(\mu) \leq \frac{1}{2}$ implies that the necessary condition for existence of an equilibrium without provision of information is violated, then the sufficient conditions in Proposition (3.3) ensures uniqueness of an equilibrium where both sellers provide information. In this context, a valid question is what types of distributions meet these sufficiency requirements. Consider the family of (increasing) convex distribution functions[8] and let $G(\cdot)$ be an element of this family. Then, by Jensen's inequality $G\left(\int_0^1 xg(x)dx\right) \leq \int_0^1 G(x)g(x)dx = \frac{1}{2}$ holds and hence, the first sufficient requirement of Proposition (3.3) is met. Second, letting $u := G(x)$, $w(u) = G^{-1}(u)$, and $w'(u) = \frac{1}{g(x)}$, where G^{-1} is the (right) inverse of G, $\Gamma(G)$ can be rewritten as follows:

$$\Gamma(G) = \int_0^1 \{u + u^4 - 2u^2\}w'(u)du$$

which, after integration by parts become equal to:

$$\Gamma(G) \quad = \quad w'(u)\lambda(z)dz\Big|_0^1 - \int_0^1 \lambda(u)w''(u)du$$

where $\lambda(s) = \int_0^s \{u + u^4 - 2u^2\}du$. It is not difficult to check that $\lambda(s) \geq 0$ for all $s \in [0,1]$. Moreover, from the implicit function theorem, $w''(u) = -\frac{g'(x)}{g^2(x)}$, $u = G(x)$, $G'(x) = g(x)$. Since $w'(u) > 0$ because we have assumed G to be increasing (hence, $g > 0$), the whole expression is nonnegative provided that $g' \geq 0$, which is precisely what the convxity of G buys. Therefore, convexity and monotonicity of G are enough to guarantee that the set of sufficient conditions in Proposition (3.3) holds true. That is, convexity and monotonicity ($g > 0$) of $G(\cdot)$ suffice to guarantee that our game features a unique equilibrium in which both sellers provide information.

[8] A distribution function G is increasing and convex if $G' = g > 0$ and $G'' = g' \geq 0$.

Corollary 3.1. *Suppose that the prior distribution of valuations F is monotone increasing and convex, i.e., $F' = f > 0$ and $F'' = f' \geq 0$. Then, the information provision game has a unique (symmetric) equilibrium such that both sellers supply information to the market.*

The most important aspect of Corollary (3.1) is to question the robustness of results in which provision of information is suboptimal from the auctioneer's point of view (Ganuza, 2004; Ganuza and Penalva, 2004; Board, 2009). Proposition (3.3) and Corollary (3.1) together show that when competition between sellers is taken into account the exact opposite may be true. For instance, if we were to assume that valuations follow a uniform distribution with support $[0, 1]$, and that only one auctioneer operates in the market, Jensen's inequality ensures that providing information always yields a price that is never higher than the price under no provision (i.e., $P^r = \mathbb{E}[\min\{v_1, v_2\}] \leq \mu = P^{nr}$). As bidders' outside option in case of no participating in the auction is exogenous and fixed, the auctioneer finds profitable to withhold information because of the higher expected price. On the contrary, when a second auctioneer is added, bidders' participation becomes endogenous and sellers can use information to affect the value of bidders' outside option. Since the uniform distribution satisfies all the requirements of Proposition (3.3), the introduction of a second auctioneer ensures the existence of an equilibrium with provision of information. Furthermore, as $f' = 0$ for the uniform distribution, Corollary (3.1) tells us that this equilibrium must be unique. Thus, with valuation uniformly distributed, competition between sellers is enough to change incentives from no provision to provision of information.

3.4 The Strategic Value of Information

Another interesting question one may ask is how the incentives to provide information changes with the strategic environments in which sellers take their decisions. In light of our previous discussion, it appears that seller's incentives to provide information are stronger in environments where one seller expects his competitor not to do so because information revelation is akin to putting a bound on the support of the distribution of valuations that this seller will face. Such control on the support of the distribution function is lost when the seller's competitior is also expected to reveal information. Moreover, in the latter case the seller is unable to avoid the presence of bidders with relatively low valuations, which is precisely what he does by revealing information againt a competitor that does not reveal. Of course, revealing information also affects the number of bidders that are expected to visit each auction. The next result shows that in the special case in which traffic is unaffected by information provision, seller's incentives to provide information decreases with the competitor's decision to do so: information behaves as if it were an strategic substitute.

Proposition 3.4. *Consider the information provision game played between sellers when bidders select trading partners using the participation rule given by* (8)*, and subsequently bid their (interim) valuations truthfully. Suppose that $F(\mu) = \frac{1}{2}$. Let ΔV be the difference between the profits accruing to seller 1 when he reveals information against a competitor who does not do so and the profits accruing to seller 1 when he reveals information against a competitor who does so. Then, $\Delta V \leq 0$*

for all $n \geq 2$.

Two comments about Proposition (3.4) are in order here. First, according to the Proposition the incentives to provide information are greater in environments where no one is providing information. A somewhat intuitive explanation for this may be as follows. When seller 1 say decides to supply information against a seller 2 who is expected not to do so, seller 1 anticipates that his action will translate into a lower bound on the set of admissible types that visit his auction. This is beneficial to seller 1 because it reduces the amount of rents he has to offer to bidders compared to what he would have to offer if all types preferred him as their trading partner. This is, seller 1 benefits from the full increase in social surplus but limits the amount of rents he has to sacrifice in order to induce truthtelling. On the other hand, providing information againts a competitor who also does so does not translate in any restriction on the support of types who prefer seller 1 over seller 2. In fact, seller 1 is no longer able to preclude the visit of *bad* customers, i.e., customers with relatively low valuations. This effect is absent in environments where only one seller supplies information and helps explain why the price seller 1 expects to receive is higher when his competitor does not provide information. Second, Proposition (3.4) simplifies the analysis by abstracting from the effect that information produces on the expected traffic. In order to achieve such simplification, we have restricted the set of admissible distribution functions by requiring that F satisfies $F(\mu) = \frac{1}{2}$. It is important to observe that this condition is weaker than symmetry of F.

The cases in which information does have an effect on traffic are much more involved because disentangling the traffic from the price effect demands additional structure to our problem. However, we are able to provide a set of necessary (sufficient) conditions for information to be an strategic complement (substitute) in market with two potential bidders.

Lemma 3.7. *Consider the information provision game played between sellers when bidders select trading partners using the participation rule given by* (8)*, and subsequently bid their (interim) valuations truthfully. Let* $\Gamma(F) = \int_0^1 \{F(x) + F^4(x) - 2F^2(x)\}dx$*, set* $n = 2$*, and define* ΔV *as in Proposition* (3.4) *Then,* $\Gamma(F) \geq 0$ *is a necessary (sufficient) condition for* $\Delta V \geq (<)0$.

4 Conclusions

In this paper we have developed a simple model intended to capture the main features of information provision in competitive environments. In our model, an equilibrium in which both sellers provide information exists under some conditions on the distribution of bidders' valuations. For markets with only two potential customers, the model allows both type of equilibria depending on whether the distribution of bidders' valuation has sufficiently low (high) values on its left (ritgh) tail. However, the equilibrium in which sellers choose uninformative structures dissapears as long as the distribution of bidders' valuations satisfies $F(\mu) \leq \frac{1}{2}$. Intuitively, $F(\mu) \leq \frac{1}{2}$ is sufficient to rule out equilibria without provision of information because distribution functions satisfying this condition tend to pure more mass on the upper end of the support, which makes more likely that bidders draw relatively high valuation and find more attractive to visit a seller who provides information rather than to visit one who does not do so. This in turn means that providing information against a competitor who does not do so attracts precisely those bidders who value the item the most allowing this seller to take advantage of the enhanced social surplus without having to offer too much much rents to his potential customers. Furthermore, we have shown that the existence of an equilibrium without provision is an artifact of the binary nature of the information structures used in the paper. When sellers are given the chance to announce any degree of informativeness they want, no equilibrium in which both sellers choose uninformative structures can exist.

We have also provided necessary and sufficient conditions under which a unique equilibrium exits such that both sellers supply information to the market. This result casts some doubts on the findings of models with a single auctioneer where providing information to markets with only two bidders is never optimal. We show that when sellers must compete for the attention of bidders, provision of information becomes the optimal choice so long as we restrict attention to distribution function satisfying some mild conditions. The reason is that with competition between sellers, bidders' participation decisions become relevant in the determination of how profitable information is because sellers can use information to affect the composition of types who visit them as well as the frequency with which each of these visits occur.

Finally, we have shown that after controlling for the effect of information on expected traffic, the incentives to provide information are weaker in environments where no one is expected to supply information. The reason for this is that provision of information against a seller who does not provide allows the seller who supplies information to keep away *bad* customer, i.e., customers whose valuations are relatively low. This effect is absent in environments where the other seller is also expected to provide information. The cases in which information affects both traffic and price are more complicated to analyze because disentangling the traffic from the price effect requires more structure in terms of the distribution of bidders' valuations and/or the number of potential bidders. We provide a set of necessary (sufficient) conditions that characterizes information as strategic complement (substitute) when the number of potential bidders is restricted to two.

Appendix

Proof of Lemma 3.1. It is straightforward to check that if bidder 1 expects everybody else to use the participation rule given in the lemma, it is a best reponse for him to use this same rule as well. Existence of a solution to equation(1) follows from the continuity of $F(\cdot)$, and from the fact that $\psi(v) \equiv vF(v)^{n-1} - \mu(1-F(v))^{n-1}$ is negative at $v = 0$ and positive at $v = 1$. Finally, uniqueness follows from $\frac{\partial \psi(v)}{\partial v} > 0$ for all $v \in (0, 1)$. □

Proof of Lemma 3.2. It suffices to prove the statement for the case $F(\mu) \leq \frac{1}{2}$ as the case where

$F(\mu) > \frac{1}{2}$ is similar. Suppose that $F(\mu) \leq \frac{1}{2}$. Let $\psi(v) := vF(v)^{n-1} - \mu(1-F(v))^{n-1}$. Then,

$$\begin{aligned} \psi(\mu) &= \mu F(\mu)^{n-1} - \mu(1-F(\mu))^{n-1} \\ &= \mu\left(F(\mu)^{n-1} - (1-F(\mu))^{n-1}\right) \\ &\leq 0 \\ &= \psi(v^*) \end{aligned}$$

and

$$\begin{aligned} \psi(v^m) &= v^m F(v^m)^{n-1} - \mu(1-F(v^m))^{n-1} \\ &= (v^m - \mu)\left(\frac{1}{2}\right)^{n-1} \\ &\geq 0 \\ &= \psi(v^*) \end{aligned}$$

because $F(\mu) \leq F(v^m) = \frac{1}{2}$ implies $\mu \leq v^m$ as $F(\cdot)$ is a strictly increasing function. Therefore, $v^* \geq \mu$ and $v^* \leq v^m$ since the function $\psi(\cdot)$ is also strictly increasing. Second, suppose that v^* satisfies $\mu \leq v^* \leq v^m$. Since $\psi(v)$ is an increasing functions of v, we must have $\psi(\mu) \leq \psi(v^*) \leq \psi(v^m)$ from where it follows that $\psi(\mu) \leq 0$. This in turn implies that the ratio $\frac{F(\mu)}{1-F(\mu)}$ is less than or equal to zero, which is equivalently to have $F(\mu) \leq \frac{1}{2}$. □

Proof of Lemma 3.3. Consider the continuation game following a history in which $p_1 = p_2 = 1$, and suppose that an equilibrium in which buyers use symmetric participation rules exists. Take any buyer with an arbitrary vector of valuations $(v_1, v_2) \in [0,1]$. After bidders have assigned themselves into the different auctions, their expected payoffs can be written using standard tools from auction theory (Myerson, 1981; Riley and Samuelson, 1981). In particular, the expected payoff a buyer with valuations (v_1, v_2) expects if she selects seller 1 with probability one can be written as:

$$R_1(v_1, v_2) = \int_{l_1}^{v_1} \tilde{Q}_1(\xi)d\xi + \max\{R_2(l_1, l_2); 0\} \tag{12}$$

where $\tilde{Q}_1(\xi) \geq 0$ is the (reduced-form) probability that any given bidder with valuation ξ wins this auction, and $(l_1, l_2) = \inf\{(v_1, v_2) : \pi(v_1, v_2) > 0\}$ is the lowest type willing to visit seller 1 with positive probability when the participation rule is given by π. Suppose that the type (v_1, v_2) finds optimal to visit seller 1 with positive probability. Then it must be true that any type $(\hat{v}_1, v_2)$, $\hat{v}_1 > v_1$ also wants to visit seller 1 with probability one because $R_1(v_1, v_2)$ is increasing in v_1 for any given any $v_2 \in [0,1]$ (Lemma 2, Myerson, 1981). Similarly, any bidder whose valuation is such that $\tilde{v}_1 < v_1$ will prefer not to visit seller 1. Therefore, for any given v_2 we can find a threshold value $\rho(v_2)$ such that all types with $v_1 \geq \rho(v_2)$ selects seller 1 with probability one. This shows existence of a function with the properties described in the lemma. Moreover, notice that ρ must satisfy $\rho(0) = 0$ because

$R_j(0,0) = 0$, $j = 1,2$, which means that $(v_1, v_2) = (0,0)$ is the lowest possible type willing to mix with positive probability between both sellers.

Next, we show that ρ is increasing. First, take any bidder and any pair of valuations (v_1, v_2) and $(v_1', \hat{v}_2)$ and suppose that $v_2 > \hat{v}_2 \Longrightarrow \rho(v_2) < \rho(\hat{v}_2)$. Take a type $(\hat{v}_1, \hat{v}_2)$ such that $\rho(v_2) < \hat{v}_1 < \rho(\hat{v}_2)$. Since $\hat{v}_1 < \rho(\hat{v}_2)$, type $(\hat{v}_1, \hat{v}_2)$ should not visit seller 1, which can happen if and only if her expected payoff when attending to this auction (where she bids truthfully) is strictly lower than what she would obtain by attending to auction 2. By construction, the type $(\rho(\hat{v}_2), \hat{v}_2)$ must be willing to mix between sellers so her expected payoffs must be equal. Let $\hat{R}$ be the expected payoff of type $(\rho(\hat{v}_2), \hat{v}_2)$. As type $(\hat{v}_1, \hat{v}_2)$ can always choose to visit seller 1 and mimic the behavior of type $(\rho(\hat{v}_2), \hat{v}_2)$ in this auction, she can always ensure an expected payoff of at least $\hat{R}$. Moreover, since $\rho(\hat{v}_2) > \hat{v}_1$ and the payoff function is increasing in v_1, $\hat{R}$ is strictly higher than what this type would get by bidding truthfully making this unilateral deviation profitable. Therefore, $\rho(\cdot)$ must be nondecreasing. Second, suppose there exists some nonempty subset $[\underline{v}_2, \bar{v}_2] \subseteq [0,1]$ where $\rho(v_2) \equiv c$ for all $v_2 \in [\underline{v}_2, \bar{v}_2]$, with $c \in (0,1)$[9]. Take any two distinct valuations $v_2', v_2'' \in [\underline{v}_2, \bar{v}_2]$ satisfying $v_2' < v_2''$, and let $R_2(v_2'') - R(v_2') = \varepsilon > 0$ (we omit v_{-j} from $R_j(\cdot)$ whenever there is no risk of confussion). By monotonicity of $R_1(\cdot)$, we can always find a $v_1' \in [0,1]$ such that $v_1' < c$ and $R_1(c) - R_1(v_1') = \varepsilon > 0$. Since $\rho(v_2'') = c$ because $v_2'' \in [\underline{v}_2, \bar{v}_2]$, a bidder with valuations (c, v_2'') should mix with positive probability between sellers, which implies that $R_1(c) - R_2(v_2'') = 0$ holds. However, as $R_2(v_2'') - R_2(v_2') = \varepsilon = R_1(c) - R_1(v_1')$ then $R_1(v_1') = R_2(v_2')$ and the type (v_1', v_2') should also be willing to mix between the sellers even though $v_1' < c = \rho(v_2')$, a contradiction. Thus, ρ must be increasing in the interval $[0,1]$.

The rest of the proof closely follows Lu (2006). By monotonicity of ρ, we can compute right and left limits at every point in the domain of ρ. Let $l = \lim_{\xi \downarrow v_2} \rho(\xi)$ and $u = \lim_{\xi \uparrow v_2} \rho(\xi)$ and suppose that ρ is discontinuous at v_2. Again, monotonicity of ρ implies that $l < u$. If $l = \rho(v_2)$, take a bidder with type (v_1, v_2) such that $v_1 \in (l, u)$. This type must expect a strictly positive payoff whenever she visits seller 1 because $v_1 > \rho(v_2)$. But then a type $(v_1, \tilde{v}_2)$ with $\tilde{v}_2$ slightly higher than v_2 should also expect a strictly positive payoff in auction 1 even though $v_1 < \rho(\tilde{v}_2)$, a contradiction. If $l < \rho(v_2) \leq u$, take a bidder whose type $(\hat{v}_1, v_2)$ is such that $l < \hat{v}_1 < \rho(v_2)$. This type should not visit seller 1 because $\hat{v}_1 < \rho(v_2)$ and so should do any type $(\hat{v}_1, \hat{v}_2)$ with $\hat{v}_2$ slightly below v_2, even though $\hat{v}_1 > \rho(\tilde{v}_2)$, a contradiction. This shows that ρ is continuous and concludes the proof. □

[9]Notice that c cannot equal 0 because this would imply that any bidder with valuations (c, v_2), $v_2 > 0$ should mix with positive probability between seller 1 and seller 2 even though $R_1(c = 0, v_2) = 0 < R_2(v_1, v_2)$. Similarly, c cannot equal one. Suppose $c = 1$. Then, by construction any type (c, v_2), $v_2 \in [\underline{v}_2, \bar{v}_2]$ is indifferent between sellers. In particular, type $(1, \underline{v}_2)$ must be indifferent between sellers and hence, $R_1(c = 1, \underline{v}_2) = R_2(v_1, \underline{v}_2)$ must hold. However, $R_2(v_1, \underline{v}_2) < R_2(v_1, \bar{v}_2)$ because $\underline{v}_2 < \bar{v}_2$, which implies that type $(c, \bar{v}_2)$ strictly prefers seller 2, a contradiction.

Proof of Lemma 3.4. The (existence part of the) proof is constructive. Set $\rho(v) = v$, $v \in [0,1]$. Then,

$$\begin{aligned} q &= 1 - \int_0^1 F(\rho(v))f(v)dv \\ & 1 - \int_0^1 F(v)f(v)dv \\ &= \frac{1}{2} \end{aligned}$$

Moreover,

$$\begin{aligned} Q_1(v) &= \left[1 - q + \int_{\{\xi:\rho(\xi)\leq v\}} \{F(v) - F(\rho(\xi))\} f(\xi)d\xi\right] \\ &= \left[\frac{1}{2} + \int_0^v \{F(v) - F(\xi)\}\, f(\xi)d\xi\right]^{n-1} \\ &= \left[\frac{1}{2} + \frac{F^2(v)}{2}\right]^{n-1} \end{aligned}$$

and,

$$\begin{aligned} Q_2(v) &= \left[q + \int_0^v F(\rho(\xi))f(\xi)d\xi\right]^{n-1} \\ &= \left[\frac{1}{2} + \int_0^v F(\xi)f(\xi)d\xi\right]^{n-1} \\ &= \left[\frac{1}{2} + \frac{F^2(v)}{2}\right]^{n-1} \end{aligned}$$

Therefore,

$$\int_0^{\rho(v)} Q_1(z)dz = \int_0^v Q_1(z)dz = \int_0^v Q_2(z)dz$$

and $\rho(v) = v$ satisfies the indifference condition (7). To show uniqueness, we make use of the following lemma.

Lemma 4.1. *Let $\delta \in (0,1)$, $0 \leq a_0 < 1$, and $0 \leq b_0 < 1$, be some fixed and known constant. Consider the following initial-value problem (IVP):*

$$\left(1 - \delta + F(z(b))F(b) - \int_{b_0}^b F(z(\xi))f(\xi)d\xi\right)^{n-1} \dot{z}(b) = \left(\delta + \int_{b_0}^b F(z(\xi))f(\xi)d\xi\right)^{n-1} \tag{13}$$

$$z(b_0) = a_0 \tag{14}$$

where F is a cdf with support $[0,1]$ and $F' = f > 0$ for all $v \in [0,1]$. Then, there exists a unique strictly increasing function $\varphi(b,\kappa)$, $\kappa = (\delta, a_0, b_0)$, defined for all $b \in [b_0, 1]$ that solves (IVP). Moreover, $\varphi(b,\kappa)$ is a continuously differentiable function of κ in the open set $D = (0,1) \times [0,1]^2$.

Proof. Consider the operator T defined by:

$$(Tz)(b) = a_0 + \int_{b_0}^{b} \gamma(z(\xi), \xi, \kappa) d\xi, \qquad b \in [b_0, 1] \tag{15}$$

where:

$$\gamma(z(\xi), \xi) = \left(\frac{\delta + \int_{b_0}^{\xi} F(z(\zeta)) f(\zeta) d\zeta}{(1-\delta) + F(z(\xi))F(\xi) - \int_{b_0}^{\xi} F(z(\zeta)) f(\zeta) d\zeta} \right)^{n-1} \tag{16}$$

To simplify notation, we suppress κ from $\gamma(z(\xi), \xi, \kappa)$ whenever there is no risk of confussion. Let $\mathcal{C} \equiv \mathcal{C}[0,1]$ be the set of continuous and nondecreasing functions defined on $[0,1]$ endowed with the sup norm. Then, $\mathcal{C}$ is a complete metric space. Take any arbitrary element in $\mathcal{C}$, say z. Because δ lives in the interior of $[0,1]$, $z \in \mathcal{C}$, and F has support $[0,1]$, $\gamma(z(\xi), \xi)$ is continuous in both arguments and $\gamma(z(\xi), \xi) > 0$ for all $\xi \in [b_0, 1]$. It follows that the operator T applied to z delivers a continuous and nondecreasing function so $(Tz) \in \mathcal{C}$. Next, it is not difficult (though tedious) to check that $\gamma(\cdot, \cdot)$ is continuously differentiable with respect to its first argument, with ξ an arbitrary element in $[b_0, 1]$. Therefore, γ is Lipschitz-continuous with respect to its first argument. Let $M > 0$ be the Lipschitz constant. For any $\epsilon > 0$ let $\varrho = \frac{\epsilon}{M}$ and take two elements z_1 and z_2 in $\mathcal{C}$ such that $\|z_1 - z_2\| \leq \varrho$. Then,

$$\begin{aligned} \|(Tz_1) - (Tz_2)\| &\leq \int_{b_0}^{b} \|\gamma(z_1(\xi), \xi) - \gamma(z_2(\xi), \xi)\| \, d\xi \\ &\leq M \int_{b_0}^{b} \|z_1 - z_2\| \, d\xi \\ &\leq \epsilon \end{aligned}$$

which shows that (Tz) is continuous. Therefore, T is a continuous operator mapping elements from $\mathcal{C}$ into elements of $\mathcal{C}$. Next, since γ is Lipschitz-continuous with respect to its first argument, the mapping $\mathcal{L}(b) = \int_0^b L(\xi) d\xi$, where:

$$L(\xi) = \int_0^{\xi} \sup_{\substack{z_1, z_2 \in \mathcal{C} \\ z_1 \neq z_2}} \left| \frac{\gamma(z_1(\zeta), \zeta) - \gamma(z_2(\zeta), \zeta)}{z_1(\zeta) - z_2(\zeta)} \right| d\zeta$$

converges to a finite number no matter what b is (so long as it is finite).

Claim 4.1. *The operator T satisfies:*

$$|(T^m z_1)(b) - (T^m z_2)(b)| \leq \frac{\mathcal{L}(b)^m}{m!} \sup_{b_0 \leq \xi \leq b} |z_1(\xi) - z_2(\xi)| \qquad z_1, z_2 \in \mathcal{C}, \; b \in [b_0, 1] \tag{17}$$

where $(T^m z)(b) = (T(T^{m-1}z))(b)$, $(T^0 z)(b) = a_0$.

Proof. Clearly, (17) holds for $m = 1$. Hence, suppose it holds for some $m > 1$. Then:

$$\begin{aligned}
|(T^{m+1}z_1)(b) - (T^{m+1}z_2)(b)]| &= \left| \int_{b_0}^{b} \{\gamma[(K^m z_1)(\xi), \xi] - \gamma[(K^m z_2)(\xi), \xi]\} \, d\xi \right| \\
&\leq \int_{b_0}^{b} |\gamma[(K^m z_1)(\xi), \xi) - \gamma[(K^m z_2)(\xi), \xi]| \, d\xi \\
&\leq \int_{b_0}^{b} L(\xi) \, |(K^m z_1)(\xi) - (K^m z_2)(\xi)| \, d\xi \\
&\leq \int_{b_0}^{b} L(\xi) \frac{\mathcal{L}(\xi)^m}{m!} \sup_{b_0 \leq \zeta \leq \xi} |z_1(\zeta) - z_2(\zeta)| \, d\xi \\
&\leq \sup_{b_0 \leq b \leq 1} |z_1(b) - z_2(b)| \int_{b_0}^{b} \frac{d\mathcal{L}(\xi)}{d\xi} \frac{\mathcal{L}(\xi)^m}{m!} d\xi \\
&= \frac{\mathcal{L}(b)^{m+1}}{(m+1)!} \sup_{b_0 \leq b \leq 1} |z_1(b) - z_2(b)|
\end{aligned}$$

where the last integral is solved by change of variables (with $u(\xi) = \frac{\mathcal{L}(\xi)^{m+1}}{(m+1)!}$). □

Let $\theta_m = \frac{\mathcal{L}(\bar{b})^m}{m!}$ such that $\sum_{j=0}^{\infty} \theta_m = \exp(\mathcal{L}(\bar{b})) < \infty$. Then, by Weissinger's fixed point theorem (Ortega and Rheinboldt, 2000) the operator T must admit a unique fixed point $\varphi \in \mathcal{C}$. Furthermore, since $\gamma(z(\xi), \xi) > 0$ the function φ must be strictly increasing (and not just nondecreasing) for all $b \in [b_0, 1]$. This shows existence of a unique increasing function that solves (IVP). To show that φ is continuously differentiable with respect to $\kappa = (\delta, a_0, b_0)$ in the set $D = (0,1) \times [0,1]^2$, notice that we have already established that $\gamma(z(\xi), \xi)$ is continuously differentiable with respect to its first argument. Some extra algebra tells us that $\frac{\partial \gamma(z(\xi),\xi)}{\partial \delta}$ and $\frac{\partial \gamma(z(\xi),\xi)}{\partial b_0}$ are also continuous function in the open set $(0,1) \times [0,1]^2$. Therefore, by Theorem 6.22 in De la Fuente (2000) $\varphi \in C^1((0,1) \times [0,1] \times [0,1])$. This completes the proof. □

As $\rho(v) = v$ solves the IVP problem of Lemma (4.1) with $a_0 = b_0 = 0$, and $\delta = \frac{1}{2}$, it follows that the function $\rho(v) = v$ must be the unique solution to Eq. (7). This completes the proof. □

Proof of Proposition 3.1. An immediate corollary of lemmas 3.1 and 3.4, and the fact that sellers are ex-ante identical whenever they announce perfectly uninformative structures ($p_1 = p_2 = 0$). □

Proof of Lemma 3.5. From lemma 3.2, if $F(\mu) \leq \frac{1}{2}$, the value of the common cutoff used by bidders to select trading partners must satisfy $\mu \leq v^* \leq v^m$. Let $F(\mu) = \frac{1}{2}$. The price expected by seller 1 when matched with k bidders is:

$$\begin{aligned}
T(k, v^*) &= \int_{v^*}^{1} v dH_{v^*}(v, k) \\
&= 4k(k-1) \int_{v^*}^{1} v(2F(v) - 1)^{k-2}(1 - F(v)) f(v) dv
\end{aligned}$$

because $F(\mu) = \frac{1}{2}$ implies that $v^* = v^m = \mu$ and hence, the distribution of types conditional on visiting seller 1 becomes:

$$\begin{aligned} G_{v^*}(v) &= \frac{F(v) - F(v^*)}{1 - F(v^*)} \\ &= 2F(v) - 1 \end{aligned}$$

It is not difficult to show that $T(k, v^*)$ is increasing in k (which holds no matter if F satisfies $F(\mu) = \frac{1}{2}$ or not). Therefore, it suffices to show that $T(2, v^*) > \mu$. Replacing $k = 2$ in the expression for $T(k, v^*)$ yields:

$$T(2, v^*) = 8 \int_{v^*}^{1} v(1 - F(v))f(v)dv$$

Let $u := F(v)$ so that $u^* = F(v^*) = \frac{1}{2}$, and $v = F^{-1}(u) := w(u)$. Then,

$$\begin{aligned} T(2, u^*) - \mu &= 8 \int_{1/2}^{1} w(u)(1 - u)du - \mu \\ &= 8 \int_{1/2}^{1} \left\{ \int_{1/2}^{u} w(z)dz \right\} du - \mu \end{aligned}$$

where the second line follows from integration by parts. As $w(u) := F^{-1}(u)$ is a strictly increasing function (because F is), $w(u) > w(1/2) = \mu$ for all $u \in (1/2, 1]$ must hold. Hence,

$$\begin{aligned} T(2, u^*) - \mu &> 8\mu \int_{1/2}^{1} (u - 1/2)\, du - \mu \\ &= 0 \end{aligned}$$

Second, let $F(\mu) < \frac{1}{2}$. Define $G_{v^*}(v)$ as the distribution of types who attend to auction 1 when the cutoff value is given by v^*:

$$G_{v^*}(v) = \frac{F(v) - F(v^*)}{1 - F(v^*)} \qquad v \in [v^*, 1]$$

and $H_{v^*}(v; k)$ denote the distribution of the second order statistics when seller 1 is matched with k bidders,

$$H_{v^*}(v, k) = G_{v^*}^{k}(v) + kG_{v^*}^{k-1}(v)(1 - G_{v^*}(v))$$

Then,

$$\begin{aligned} \frac{\partial H_{v^*}(v, k)}{\partial v^*} &= k(k-1)G_{v^*}^{k-2}(v)(1 - G_{v^*}(v)) \frac{\partial G_{v^*}(v)}{\partial v^*} \\ &= k(k-1)G_{v^*}^{k-2}(v)(1 - G_{v^*}(v)) \frac{-f(v^*)(1 - F(v))}{(1 - F(v^*))^2} \\ &< 0 \end{aligned}$$

for all $k \geq 2$, and $H_{v^*}(v, k)$ is strictly increasing in the cutoff value v^*. Therefore, $H_a(v; k)$ first-order stochastically dominates $H_b(v; k)$ provided that $a > b$.

Claim 4.1. Let $\alpha = F(\mu) < 1/2$ and v^*_α be the solution to $v^*_\alpha \alpha^{n-1} = \mu(1-\alpha)^{n-1}$. Then, $v^*_\alpha > \mu$.

Proof. Rewrite v^*_α as:

$$\mu = v^*_\alpha \left(\frac{\alpha}{1-\alpha}\right)^{n-1}$$

and suppose that $v^*_\alpha > \mu$. Since the term in brackets is strictly increasing in α, $\left(\frac{\alpha}{1-\alpha}\right)^{n-1} < 1$. If $v^*_\alpha \leq \mu$ then,

$$\mu = v^*_\alpha \left(\frac{\alpha}{1-\alpha}\right)^{n-1} < \mu$$

a contradiction. □

From the above claim, it follows that the cutoff value when F satisfies $F(\mu) < \frac{1}{2}$ is greater than the cutoff when it satisfies $F(\mu) = \frac{1}{2}$. Therefore, the expected price when $F(\mu) < \frac{1}{2}$ must be greater than the price when $F(\mu) = \frac{1}{2}$ and hence, $T(k, v^*) > \mu$ holds for all $k = 2, \ldots, n$ if $F(\mu) \leq \frac{1}{2}$. □

Proof of Proposition 3.2. To show necessity, first note that lemmas 3.5 and 3.2 ensure that whenever $F(\mu) \leq \frac{1}{2}$ then $V_1(1,0) > V_1(0,0)$ for all $n \geq 2$ because both the expected price and the probability with which any given visits seller 1 when $p_1 = 1$ and $p_2 = 0$ cannot be less than $\frac{1}{2}$. Therefore, if $V_1(1,0) \leq V_1(0,0)$ holds for some $\hat{n}$, then $F(\mu) > \frac{1}{2}$ must be true.

To show sufficiency, set $n = 2$ and suppose that $V_1(1,0) > V_1(0,0)$ holds. Let $G_{v^*}(v) = \frac{F(x)-F(v^*)}{1-F(v^*)}$ be the distribution of valuations of bidders participating in auction 1, with v^* the cutoff value employed by bidders to select trading partners. Then, $H_{v^*}(x) = G^2_{v^*}(x) + 2G_{v^*}(x)(1 - G_{v^*}(x))$ is the probability distribution of the second highest valuation of bidders attending to auction 1. It is straightforward to check that $H_{v^*}(x)$ first order stochastically dominates $G_{v^*}(x)$ and hence, that $\int_{v^*}^1 v dH_{v^*}(v) \leq \int_{v^*}^1 v dG_{v^*}(v)$ holds (Ganuza and Penalva, 2006). Therefore,

$$\begin{aligned}
V_1(1,0) - V_1(0,0) &= (1-F(v^*))^2 \int_{v^*}^1 v dH_{v^*}(v) - \left(\frac{1}{4}\right)\mu \\
&\leq (1-F(v^*))^2 \int_{v^*}^1 v dG_{v^*}(v) - \left(\frac{1}{4}\right)\mu \\
&= (1-F(v^*))^2 \left(\frac{\int_{v^*}^1 v f(v) dv}{1-F(v^*)}\right) - \left(\frac{1}{4}\right)\mu \\
&\leq (1-F(v^*)) \int_0^1 v f(v) dv - \left(\frac{1}{4}\right)\mu \\
&= \left(\frac{3}{4} - F(v^*)\right)\mu
\end{aligned}$$

from where it follows that $V_1(1,0) > V_1(0,0)$ implies $F(v^*) < \frac{3}{4}$. Therefore, $F(v^*) \geq \frac{3}{4}$ is sufficient for the existence of some n ($n = 2$) such that there is an equilibrium where sellers do not provide information. □

Proof of Lemma 3.6. Let $\tilde{H}(x)$ be the distribution of the second order statistics when the underlying distribution is $F^2(x)$ and $n = 2$, i.e., $\tilde{H}(x) = F^4(x) + 2F^2(x)(1 - F(x))$. Then,

$$\begin{aligned}
\tilde{T} - \mu &= \int_0^1 x d\tilde{H}(x) - \int_0^1 x dF(x) && (18)\\
&= \int_0^1 \left\{ F(x) - \tilde{H}(x) \right\} dx \\
&= \int_0^1 \left\{ F(x) + F^4(x) - 2F^2(x) \right\} dx && (19)\\
&= \Gamma(F) && (20)
\end{aligned}$$

where the second line follows from integration by parts. It is obvious that $\Gamma(F) \geq (<) 0$ if and only if $\tilde{T} \geq (<) \mu$. □

Proof of Lemma **??**. Let $\tilde{H}(x, k)$ be the distribution of the second order statistics when the underlying distribution function is $\tilde{G}(x) = F^2(x)$, and seller 1 is matched with exactly $k = 2, \cdots, n$ bidders. Since $\tilde{H}(x, k)$ first-order stochastically dominates $\tilde{H}(x, k-1)$ for all $k = 3, \cdots, n$, showing that $\tilde{H}(x, 3)$ stochastically dominates $F(x)$ suffices to prove the claim. Let $\phi(x) := F(x) - \tilde{H}(x, 3)$. Simple algebraic manipulation yields:

$$\begin{aligned}
\phi(x) &= F(x) - \left[F^6(x) + 3F^4(x)(1 - F^2(x)) \right] \\
&= F(x) - 3F^4(x) + 2F^6(x)
\end{aligned}$$

where the first line follows from the definition of $\tilde{H}(x, 3)$ and $\tilde{G}(x) = F^2(x)$. It is straightforward to check that $\phi(x)$ is nonnegative for all $x \in [0, 1]$. Therefore, $F(x) \geq \tilde{H}(x, 3)$ and $\tilde{H}(x, 3)$ first-order stochastically dominates $F(x)$. By Theorem 7 in Ganuza and Penalva (2004), the expected value of a random variable distributed according to $\tilde{H}(x, 3)$ is never less than the expected value of a random variable distributed according to $F(x)$, and hence $\tilde{T}(k) \geq \mu$ must hold for $k = 3$, as desired. □

Proof of Proposition 3.3.

(i) Necessity. We prove the contrapositive statement, i.e, we prove that if $F(\mu) > \frac{1}{2}$ and $\Gamma(F) < 0$ then $V_1(0, 1) > V_1(1, 1)$ holds for $n = 2$. Let $\tilde{T}$ be the expected price prevailing in auction 1 when $p_1 = p_2 = 1$ and seller 1 is matched with exactly two bidders. From lemma 3.6 $\Gamma(F) < 0$ if and only if $\tilde{T} < \mu$. Moreover, from Proposition (3.1) any given bidder visits seller 1 with probability $\frac{1}{2}$ whenever $p_1 = p_2 = 1$, and visits seller 1 with probability $F(v^*)$ whenever $p_1 = 0$

and $p_1 = 1$, where v^* is the solution to Eq. (1). Therefore,

$$\begin{aligned} V_1(0,1) - V_1(1,1) &= F^2(v^*)\mu - \left(\frac{1}{2}\right)^2 \tilde{T} \\ &> \left[F^2(v^*) - \left(\frac{1}{2}\right)^2\right]\mu \\ &\geq 0 \end{aligned}$$

where the first inequality follows from $\tilde{T} < \mu$, and the second inequality follows from Lemma 3.2 and the fact that $F(v^*) > \frac{1}{2}$ if and only if $F(\mu) > \frac{1}{2}$.

(ii) Sufficiency. From the expressions for $V_1(1,1)$ and $V_1(1,0)$ in the text we have:

$$V_1(1,1) - V_1(0,1) = \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n \tilde{T}(k) - \sum_{k=2}^{n} \binom{n}{k} [F(v^*)]^k [1 - F(v^*)]^{n-k} \mu$$

If $\Gamma(F) \geq 0$ holds then lemma 3.6 ensures that $\tilde{T}(k) \geq \mu$ for all $k \geq 2$. Therefore,

$$\begin{aligned} V_1(1,1) &= \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n \tilde{T}(k) \\ &\geq \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n \mu \\ &= \mu \left[1 - (1+n)\left(\frac{1}{2}\right)^n\right] \end{aligned}$$

Define $\zeta(y,n)$ as follows:

$$\zeta(y,n) = ny(1-y)^{n-1} + (1-y)^n - (n+1)\left(\frac{1}{2}\right)^n$$

This expression is nonnegative for all $n \geq 2$ provided that $y \leq \frac{1}{2}$. Hence, if $\Gamma(F) \geq 0$ and $y \leq \frac{1}{2}$ we have:

$$\begin{aligned} V_1(1,1) &\geq \mu \left[1 - (1+n)\left(\frac{1}{2}\right)^n\right] \\ &\geq \mu \left[1 - (1-y)^n - ny(1-y)^{n-1}\right] \\ &= \sum_{k=2}^{n} \binom{n}{k} y^k (1-y)^{n-k} \mu \end{aligned}$$

From lemma 3.2 a necessary and sufficient condition for $F(v^*) \leq \frac{1}{2}$ to hold is $F(\mu) \leq \frac{1}{2}$. Let $y = F(v^*)$. With this substitution, the last line in the previous expression becomes equal to $V_1(0,1)$, and hence $V_1(1,1) \geq V_1(0,1)$ must hold true for all $n \geq 2$ if $\Gamma(F) \geq 0$ and $F(\mu) \leq \frac{1}{2}$. □

Proof of Corollary 3.1. Existence of an equilibrium in which both sellers reveal information follows from the discussion in the main text. Morever, as Jensen's inequality ensures that $F(\mu) \leq \frac{1}{2}$ holds for F convex, the necessary condition for the existence of an equilibrium without revelation given by Proposition 3.2 is violated. Therefore, the game can have at most one equilibrium within the class of symmetric Perfect Bayesian equilibria. □

Proof of Proposition 3.4. Without loss of generality, consider seller 1. Suppose seller 2 sets $p_2 = 0$ and let ΔV^{nr} be the difference in seller 1's profits when he sets $p_1 = 1$ and $p_1 = 0$ given that $p_2 = 0$:

$$\Delta V^{nr} = V_1(1,0) - V_1(0,0)$$

Similarly, define ΔV^{r} be the difference in seller 1's profits when he sets $p_1 = 1$ and $p_1 = 0$ given that seller 2 has chosen $p_2 = 1$:

$$\Delta V^{r} = V_1(1,1) - V_1(0,1)$$

We want to establish the sign of $\Delta V := \Delta V_1^r - \Delta V_1^{nr}$. Simple algebraic manipulation yields:

$$\begin{aligned}\Delta V &= \Delta V_1^r - \Delta V_1^{nr} \\ &= [V_1(1,1) - V_1(0,1)] - [V_1(1,0) - V_1(0,0)] \\ &= [V_1(1,1) - V_1(1,0)] - [V_1(0,1) - V_1(0,0)]\end{aligned}$$

Let $\tilde{H}(v,k)$ be distribution of the second order statistics when the underlying distribution is equal to $F^2(v)$ and the number of bidders is equal to $k \geq 2$. Similarly, let $H_{v^*}(v,k)$ be distribution of the second order statistics when the underlying distribution is $\frac{F(v)-F(v^*)}{1-F(v^*)}$, with v^* the cutoff value given by Eq. (1). Define $\Phi(v,k)$ as follows:

$$\Phi(v,k) := \begin{cases} -\tilde{H}(v,k) & \text{if } v < v^* \\ H_{v^*}(v,k) - \tilde{H}(v;k) & \text{if } v \geq v^* \end{cases}$$

Let $u := F(v)$. Hence,

$$\tilde{H}(u,k) = u^{2k} + k u^{2(k-1)}(1-u^2)$$

and

$$\begin{aligned} H_{v^*}(u,k) &= \left(\frac{u-u^*}{1-u^*}\right)^k + k\left(\frac{u-u^*}{1-u^*}\right)^{k-1}\left[1 - \left(\frac{u-u^*}{1-u^*}\right)\right] \\ &= (2u-1)^k + k(2u-1)^{k-1}(1-(2u-1)) \end{aligned}$$

because lemma 3.2 ensures that $u^* = F(v^*) = \frac{1}{2}$ if and only if $F(\mu) = \frac{1}{2}$. Therefore, $\Phi(\cdot,2)$ can be rewritten as follows:

$$\Phi(u,k) := \begin{cases} (k-1)u^{2k} - k u^{2k-2} & \text{if } u < 1/2 \\ k\left[(2u-1)^{k-1} - u^{2k-2}\right] - (k-1)\left[(2u-1)^k - u^{2k}\right] & \text{if } u \geq 1/2 \end{cases}$$

Some algebra shows that $\Phi(u,2) \leq 0$ holds for all $u \in [0,1]$ and all $k \geq 2$. Since $F(v)$ is a strictly increasing function of v it must be the case that $\Phi(v,2) \leq 0 \; \forall \, v \in [0,1]$ also holds. We conclude that $H_{v^*}(v,k)$ first-order stochastically dominates $\tilde{H}(v,k)$ for all $k \geq 2$. This implies that the expected price when seller 1 and 2 both announce $(p_1 = 1, p_2 = 0)$ must be greater than or equal to the expected price that seller 1 receives when $(p_1 = 1, p_2 = 1)$. Since we have assumed that $F(\mu) = \frac{1}{2}$ and thus, $F(v^*) = \frac{1}{2}$, the visiting probabilities when $(p_1 = 1, p_2 = 0)$ and $(p_1 = p_2 = 1)$ are the same. Therefore, $V_1(1,1) - V_1(1,0) \leq 0$ and, $V_1(0,1) - V_1(0,0) = 0$ because $F(v^*) = \frac{1}{2}$ and the price when $p_1 = 0$ and $p_2 = 1$ is the same as the one when $p_1 = 0$ and $p_2 = 0$. It follows that $\Delta V \leq 0$ for all $n \geq 2$ and therefore, information behaves as an strategic substitute whenever $F(\mu) = \frac{1}{2}$. □

Proof of Lemma 3.7. Set $n = 2$. By lemma 3.5 the expected price when seller 1 announces $p_1 = 1$ and seller 2 announces $p_2 = 0$, $\tilde{T}$, is greater than μ for all $n \geq 2$. Therefore,

$$
\begin{aligned}
V_1(1,0) - V_1(0,0) &= (1 - F(v^*))^2 \tilde{T} - \left(\frac{1}{4}\right) \mu \\
&\geq \left\{ (1 - F(v^*))^2 - \left(\frac{1}{4}\right) \right\} \mu
\end{aligned}
$$

Next, from lemma 3.6 the condition $\Gamma(F) < 0$ implies that the expected price when $p_1 = p_2 = 1$ and $n = 2$, $\hat{T}$, is strictly less than μ. Therefore,

$$
\begin{aligned}
V_1(1,1) - V_1(0,1) &= \left(\frac{1}{4}\right) \hat{T} - F^2(v^*) \mu \\
&< \left\{ \left(\frac{1}{4}\right) - F^2(v^*) \right\} \mu
\end{aligned}
$$

and,

$$
\begin{aligned}
\Delta V_1 &= [V_1(1,1) - V_1(0,1)] - [V_1(1,0) - V_1(0,0)] \\
&< \left\{ \left(\frac{1}{4}\right) - F^2(v^*) \right\} \mu - \left\{ (1 - F(v^*))^2 - \left(\frac{1}{4}\right) \right\} \mu \\
&= \left\{ \frac{1}{2} - (1 - F(v^*))^2 - F^2(v^*) \right\} \mu \\
&\leq 0
\end{aligned}
$$

because the term within brackets is non positive for all $x \in [0,1]$. Therefore, if $\Gamma(F) < 0$ then $\Delta V < 0$ from where it follows that $\Gamma(F) \geq 0$ is necessary for $\Delta V \geq 0$. □

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