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Subsidies for whom? The rule of (G+1)/2

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Abstract

This paper shows that for a society with inequality described by a Gini coefficient G, a pivotal individual is located in the $(G+1)/2$ percentile: any dollar given to an individual poorer (richer) than himself decreases (increases) G. As a consequence, an optimal lump sum subsidy, in terms of the Gini index, is the one given to all individuals ranked below this percentile. We show that in 2/3 of the countries such an individual is within the eighth decile. Hence, from a comparative perspective, the $(G+1)/2$ percentile is the *rule of the eighth decile*. All subsidies targeted below this decile contribute to egalitarian redistribution. This result is robust to the use of other inequality measures, such as Theil indices.

1 Introduction

One of the main objectives of redistribution is to reduce inequality. Egalitarian redistribution is achieved through subsidies to the poorer members of society. While it is obvious that a dollar given to the very poor decreases inequality, the identity of the richest person who can receive the subsidy without reverting its intended egalitarian effect is not so clear. The identity of such pivotal individual is primarily a theoretical problem, but it has practical importance as well. An optimal lump sum subsidy, in terms of equality, is the one given to every member of society whose contribution to inequality is negative after the transfer, meaning all individuals poorer than the pivotal agent. Therefore this paper addresses the question about how many deciles need to be subsidized - transfers, education or health insurance - in order to reduce inequality.

This paper shows that for a society with inequality described by a Gini coefficient G, a pivotal individual is located in the $(G+1)/2$ percentile: any dollar given to an individual poorer (richer) than himself decreases (increases) the Gini coefficient. As a consequence, an optimal lump sum subsidy, in terms of the Gini index, is the one given to all individuals ranked below this percentile.

As the Gini coefficient is the most popular index of inequality for cross country comparisons, our rule allows a simple assessment of the optimal percentile for a lump sum subsidy in a variety of countries. The results are remarkably general given the mild variance of the Gini coefficients across the world: about 2/3 of all the countries have the pivotal individual within the eighth decile of the income distribution. Hence from a comparative perspective, the (G+1)/2 is the *rule of the eighth decile*. All subsidies targeted below this decile contribute to egalitarian redistribution.

Finally, we consider the fact that many results within the inequality literature, such as the cross-country rankings of inequality, depend on the measure used (Atkinson 1970). Thus, we compute the identity of the pivotal individual under alternative measures. In the case of Theil 0 and Theil 1 indices, we found that the new pivots are in a range of 10% around the Gini percentile. While the rule exhibits some sensitivity to the indices, the changes are small enough to assure that the $(G+1)/2$ rule provides a useful guide for egalitarian redistribution.

2 The Rule of $(G+1)/2$

Consider a population of n individuals whose incomes are given by y_i. Individuals are ranked in i such that $y_i \leq y_{i+1}$. An individual ranked j in the distribution of income occupies the percentile $p_j = j/n$.

We define the Gini index G as a function of ordered incomes (Sen 1973, Dasgupta et al 1973)

$$G = \frac{2}{\mu n^2} \sum\nolimits_{i=1}^{n} i y_i - \frac{n+1}{n}$$

where μ is the mean income. Henceforth we replace $(n+1)/n$ by 1 under the assumption that n is large enough to neglect n^{-1}.

First we study the change in the Gini coefficient when the individual in the percentile p_j is subsidized with one dollar. We assume that the dollar is collected through a proportional tax charged to all individuals, a mechanism that has no effect over the Gini index (Dalton 1920)[1]. We called G the initial Gini, $G(p_j)$ the Gini after the subsidy and $\Delta G(p_j) = G(p_j) - G$ the change in the Gini index. Note that $\Delta G(1) > 0$ because a subsidy given to the richest individual in society increases inequality. On the contrary, a one-dollar subsidy to the poorest

[1]Propostion 1 does not require the assumption of proportional taxation; it still holds when the subsidy is not collected from the individuals.

member decreases the Gini index, and thus $\Delta G(0) < 0$. Therefore we conjecture the existence of p^* such that $\Delta G(p^*) = 0$. The following proposition characterizes $\Delta G(p_j)$ and p^* for the general case in which the subsidy can be large enough to change the income ranking across individuals.

Proposition 1 *A subsidy c, collected through a proportional income tax, is given to an individual in the percentile p_j with income y_j. Let $p_j + \delta_j(c)$ be the percentile of the individual with income $y_j + c$. Then $\Delta G(p_j)$ satisfies*

$$(p_j - p^*) \frac{2c}{\mu n} \leq \Delta G(p_j) \leq (p_j + \delta_j(c) - p^*) \frac{2c}{\mu n}$$

with

$$p^* = \left(\frac{G+1}{2} \right)$$

Proof. Appendix. ■

From Proposition 1, we notice that if $p_j > p^*$, then $\Delta G(p_j) > 0$ and if $p_j < p^* - \delta_j(c)$, then $\Delta G(p_j) < 0$. Accordingly, the percentile p_j that satisfies $\Delta G(p_j) = 0$ falls within the interval $(p^* - \delta_j(c), p^*)$. Thus, the question is about the magnitude of $\delta_j(c)$. Given that a Gini of 40 has an associated $p^* = 70$, we evaluate $\delta_j(c)$ close to that percentile. For that purpose, we use the distribution of income of US households (US Census 2010). Households with annual incomes of $\$60,000$, $\$70,000$ and $\$80,000$ are in the 58, 65 and 71 percentiles, respectively. Accordingly, a subsidy as large as $\$10,000$ generates a δ_j between 6 and 7%. A more reasonable $\$1,000$ subsidy generates a δ_j close to 1%. Therefore, we conclude that for subsidies as large as $\$1,000$, the magnitude of $\delta_j(c)$ is negligible. If that is the case, Proposition 1 simplifies to the following rule:

Corollary 2 (the rule of $(G+1)/2$) *Assume that the subsidy c in Proposition 1 is small enough to preserve the income ranking across individuals. Then the individual located in the percentile $p^* = (G+1)/2$ of the income distribution is pivotal in the following sense*

$$\begin{aligned}
\Delta G(p_j) &< 0 \text{ if } (p_j < p^*) \\
\Delta G(p_j) &= 0 \text{ if } (p_j = p^*) \\
\Delta G(p_j) &> 0 \text{ if } (p_j > p^*)
\end{aligned}$$

Proof. From Proposition 1, $\delta_j(c) = 0$ implies $\Delta G(p_j) = (p_j - p^*) \, 2c/\mu n$ ■

Secondly, we consider the following problem: how does the Gini change if a lump sum subsidy c is given to all individuals ranked below a percentile p_j? Let $G(0, ..., p_j)$ be the Gini after the subsidy, and $\Delta G(0, ..., p_j) = G(0, ..., p_j) - G$. To characterize $\Delta G(0, ..., p_j)$, we assume[2] that the revenue is $cj = c \times j$ and each of the poorer j members of the population receives a subsidy c.

Proposition 3 *Suppose a transfer cj is collected through a proportional income tax and c is given to all individuals from percentile 0 to percentile j. Assuming that the transfers do not change the ranking of incomes, then the change in the Gini index $\Delta G(0, ..., p_j)$ is given by*

$$\Delta G(0, ..., p_j) = \left(\frac{1}{2} p_j^2 - p_j p^* \right) \frac{2c}{\mu}$$

and thus the minimum is achieved by $p_j = p^*$.

Proof. Appendix. ■

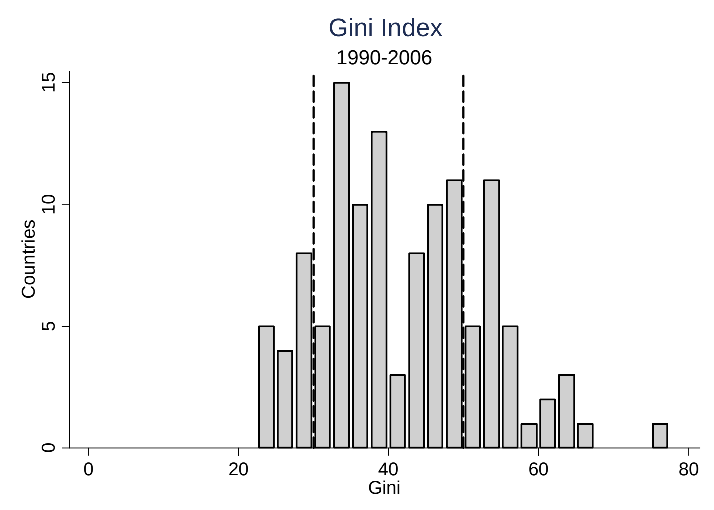
Proposition 3 shows that the optimal lump sum subsidy, in terms of reducing the Gini coefficient, is the one targeted at all individuals poorer than the pivotal individual located in $p^* = (G + 1)/2$ of the income distribution. The next section discusses this result in the context of cross-country inequality.

3 The rule around the world

To illustrate our rule we use the World Income Inequality Database (WIID 2.0, 2009) from the United Nations. The data set includes both the Gini coefficient and the income shares disaggregated by deciles, for a variety of countries and years. Instead of using reported Gini indices, we prefer to compute the Gini over income deciles since we will use those deciles to compare our rule with other inequality measures next. In any case, the correlation for the entire sample is above 98 per cent. For the calculation done over deciles, the $(G + 1)/2$ formula should be slightly modified since n^{-1} is non negligible for $n = 10$. Accordingly, we add 5% to the optimal percentile.

[2] This assumption means that taxation cannot be separated from the total subsidy. If the revenue is equal to c with independence of who receives the subsidy, then the optimal transfer is trivial: all of c is given to the poorest member of the distribution.

Figure 1 displays the histogram of the Gini coefficient (simple averages for all available data since 1990) for 121 countries around the world



The cross-country average for the Gini index is 42 and the median is 41. According to Propositions 1 and 2, the associated p^* is 76%. Compare to other economic variables, the Gini coefficient has a smaller cross-country variance; therefore, a large fraction of countries are remarkably close to the optimal percentile of the mean. Countries with Gini indices between 30 and 50, such as the ones between the two dotted lines drawn in Figure 1, have the pivotal percentile in the eighth decile of the distribution, namely between 70 and 80% Seventy five out of 121 countries, or 62% of the sample, fall within this range.From a worldwide perspective, the $(G+1)/2$ is the *rule of the eighth decile.*

So far we have focused our analysis on the Gini coefficient, the more popular index of inequality and the more appropriate for cross-country comparisons. However, country rankings depend on how inequality is measured (Atkinson 1970). This lack of consistency can be problematic in our context if, under another measurement, the optimal percentile does not remain close to the eighth decile but, for instance, it decreases to 40%.

In order to check the robustness of the rule, we consider the Theil indices (Theil 1967). This family of indices is closely related to a number of other commonly used families of inequality measures (Cowell 2006). In particular, we consider the so-called Theil 0 and Theil 1 indices, which can also be seen as transformations of the Atkinson indices. Theil indices are defined as follows

$$T_0 = \frac{1}{n}\sum\nolimits_{i=1}^{n} \ln\left(\frac{\mu}{y_i}\right)$$
$$T_1 = \frac{1}{n}\sum\nolimits_{i=1}^{n} \left(\frac{y_i}{\mu}\right)\ln\left(\frac{y_i}{\mu}\right)$$

We solve the problem of finding the pivotal individual of the distribution for the case of Theil indices. However, the results are conceptually different. While a transfer from individual i to j changes the Gini based upon the differences in the ranking between i and j, it changes the Theil according to the differences between y_i and y_j (Allison 1976). Hence the equivalent to Proposition 1 for the Theil index will be stated in terms of an optimal income y^* instead of an optimal percentile p^*. To get a simple expression we follow Allison (1976) and assume that the transfer is very small (one dollar transfer). This limit argument is consistent with the ranking preserving assumption used to derive Corollary 2.

Proposition 4 *Assume that a transfer c is collected through a proportional income tax and given to an individual with income y_j. Further assume that c is small enough to consider only terms at first order in c. Then inequality decreases, $\Delta T_{0,1} < 0$, if and only if $y_j < y^*_{0,1}$ with*

$$y_0^* = \mu$$
$$y_1^* = \mu \exp(T_1)$$

Proof. Appendix. ■

We use Proposition 4 to compare optimal subsidies using the Theil index against the rule of $(G+1)/2$. For each country we compute G, y_0 and y_1 using WIID data for deciles. First, we compute "Gini Percentile" as $(G+1)/2$ adjusted for deciles. Second, we identify the decile where y_0 and y_1 are placed, and then we use a linear interpolation of the same decile data to find the "Theil Percentile" for both Theil0 and Theil1. Figure 2 displays our results.

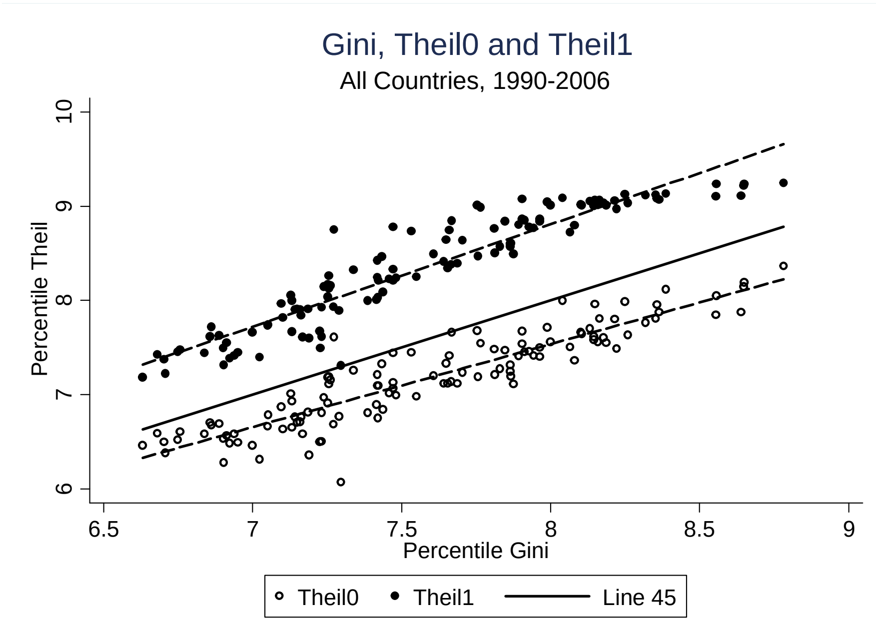


Figure 2 shows that pivotal percentiles are different for each inequality index but these differences are not dramatic. For cross-country averages, percentiles for the Gini, Theil0, and Theil1 are 76, 73 and 83 percent, respectively. The Theil percentiles are in a range of 10% around $(G+1)/2$. We conclude that although percentiles exhibit some sensitivity to the indices, the changes are small enough to assure that the $(G+1)/2$ rule provides a useful guide for egalitarian redistribution.

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4 Appendix

Proof Proposition1. Individuals are taxed in a proportion τ of their income. The total revenue $\tau\mu n$ is equal to c, which gives $\tau = c/\mu n$. Let $\widetilde{y}_j$ be the post tax income, i.e., $\widetilde{y}_j = y_j(1 - c/n\mu)$.

Consider the individual j who receives a subsidy c. As a result, he improves his position in the ranking of income to a position k, with $k > j$. The following relation determines k

$$(\widetilde{y}_k - \widetilde{y}_j) \leq c < (\widetilde{y}_{k+1} - \widetilde{y}_j)$$

before the Gini computation, we compute the sum of iy_i when j receives c and occupies the ranking k

$$\sum_{i=1}^{j-1} i\widetilde{y}_i + \sum_{i=j+1}^{k-1} (i-1)\widetilde{y}_i + k(\widetilde{y}_j + c) + \sum_{i=k+1}^{n} i\widetilde{y}_i = \sum_{i=1}^{n} i\widetilde{y}_i - \sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j) + kc$$

Now we compute the Gini

$$\begin{aligned}
G(p_j) &= \frac{2}{\mu n^2}\left(\sum_{i=1}^{n} iy_i \left(1 - \frac{c}{n\mu}\right) - \sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j) + kc\right) - 1 \\
&= G + \frac{2}{\mu n^2}\left(-\frac{c}{n\mu}\sum_{i=1}^{n} iy_i - \sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j) + kc\right) \\
&= G + \frac{2c}{\mu n}\left(\frac{k}{n} - \frac{G+1}{2}\right) - \frac{2}{\mu n^2}\sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j) \\
&= G + \frac{2c}{\mu n}\left(\frac{j}{n} - \frac{G+1}{2}\right) - \frac{2}{\mu n^2}\left(\sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j - c)\right)
\end{aligned}$$

In the last equation, the last term is negative since $\widetilde{y}_i > \widetilde{y}_j$, $\forall i = j+1, ...k-1$. We write $k/m = p_k = p_j + \delta_j(c)$ and so $\Delta G(p_j) - (p_j + \delta_j(c) - p^*)\, 2c/\mu n \leq 0$.

We rewrite the last expression as

$$G(p_j) = G + \frac{2c}{\mu n}\left(\frac{j}{n} - \frac{G+1}{2}\right) - \frac{2}{\mu n^2}\left(\sum_{i=j+1}^{k-1} (\widetilde{y}_i - \widetilde{y}_j - c)\right)$$

Now the last term is positive. According to the definition of k, $(\widetilde{y}_k - \widetilde{y}_j - c) \leq 0$ and $\widetilde{y}_i < \widetilde{y}_k$, $\forall i = j+1, ...k-1$. We conclude $\Delta G(p_j) - (p_j - p^*)\, 2c/\mu n \geq 0$.■

Proof Proposition3. As the revenue is cj, proportional taxation is equal to $\tau = cj/\mu n$.

The Gini is

$$
\begin{aligned}
G(p_j) &= \frac{2}{\mu n^2}\left(\sum\nolimits_{i=1}^{n} i y_i \left(1-\frac{cj}{n\mu}\right)+\sum\nolimits_{i=1}^{j} ic\right)-1 \\
&= G+\frac{2c}{\mu}\left(\frac{j^2}{2n^2}-\frac{j}{n^3\mu}\sum\nolimits_{i=1}^{n} i y_i\right) \\
&= G+\frac{2c}{\mu}\left(\frac{1}{2}p_j^2-p_j\left(\frac{G+1}{2}\right)\right)
\end{aligned}
$$

Proof Proposition4. We compute the first order expansion of T_0 and T_1 in terms of c We define $\delta_{ij}=1$ if $i=j$ and 0 otherwise

$$
\begin{aligned}
T_0(y_j) &= \ln(\mu)-\frac{1}{n}\sum\nolimits_{i=1}^{n}\ln\left(y_i\left(1-\frac{c}{n\mu}\right)+c\delta_{ij}\right) \\
&= T_0+\frac{1}{n}\sum\nolimits_{i=1}^{n}\left(\frac{1}{y_i}\left(\frac{y_i}{n\mu}-\delta_{ij}\right)\right)c+O(c^2) \\
&= T_0+\left(1-\frac{\mu}{y_j}\right)\frac{c}{n\mu}+O(c^2)
\end{aligned}
$$

and the same for $T_1(y_j)$

$$
\begin{aligned}
T_1(y_j) &= T_1+\frac{1}{\mu n}\sum\nolimits_{i=1}^{n}\left(\left(-\frac{y_i}{n\mu}+\delta_{ij}\right)\ln\left(\frac{y_i}{\mu}\right)+y_i\frac{\mu}{y_i}\left(-\frac{y_i}{n\mu}+\delta_{ij}\right)\right)c+O(c^2) \\
&= T_1+\frac{1}{\mu n}\left(-\frac{1}{n}\sum\nolimits_{i=1}^{n}\frac{y_i}{\mu}\ln\left(\frac{y_i}{\mu}\right)+\ln\left(\frac{y_j}{\mu}\right)-\mu+\mu\right)c+O(c^2) \\
&= T_1+\frac{1}{\mu n}\left(\ln\left(\frac{y_j}{\mu}\right)-T_1\right)c+O(c^2)
\end{aligned}
$$